Some properties of the noncommutative H p (r , s) (; ℓ ∞) and H p (; ℓ 1) spaces

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Some properties of the noncommutative $H_p^{(r,s)}(\mathscr{A};\ell_\infty)$ and $H_p(\mathscr{A};\ell_1)$ spaces

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Abstract. In this paper, we introduce the noncommutative $H_p^{(r,s)}(\mathscr{A};\ell_\infty)$ and $H_p(\mathscr{A};\ell_1)$ spaces. Then, it is shown that both spaces are Banach spaces for $r,s \geq 2$ (and resp. $p \geq 1$) and the analogue of Saito's theorem for the $H_p^{(r,s)}(\mathscr{A};\ell_\infty)$ and $H_p(\mathscr{A};\ell_1)$ spaces are proved.

Keywords: Von Neumann algebra, Subdiagonal algebras, Vector valued noncommutative Hardy spaces
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INTRODUCTION

Let H be a Hilbert space and $\mathscr{M}$ be a finite von Neumann algebra equipped with a normal faithful tracial state τ [12, 13, 16]. Let $\mathscr{D}$ be a von Neumann subalgebra of $\mathscr{M}$, and let $\Phi : \mathscr{M} \to \mathscr{D}$ be the unique normal faithful conditional expectation such that $\tau \circ \Phi = \tau$. A w^*-closed subalgebra $\mathscr{A}$ of $\mathscr{M}$ is called finite subdiagonal algebra of $\mathscr{M}$ with respect to Φ, if it is satisfying the following conditions:

(i) $\mathscr{A} + \mathscr{A}^*$ is w^*-dense in $\mathscr{M}$;
(ii) Φ is multiplicative on $\mathscr{A}$, i.e., $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in \mathscr{A}$;
(iii) $\mathscr{A} \cap \mathscr{A}^* = \mathscr{D}$, where $\mathscr{A}^*$ is the family of all adjoint elements of the element of $\mathscr{A}$, i.e., $\mathscr{A}^* = \{a^* : a \in \mathscr{A}\}$.

The algebra $\mathscr{D}$ is called the diagonal of $\mathscr{A}$. It's proved by Exel [10] that a finite subdiagonal algebra $\mathscr{A}$ is automatically maximal in the sense that if $\mathscr{B}$ is another subdiagonal algebra with respect to Φ containing $\mathscr{A}$, then $\mathscr{B} = \mathscr{A}$. This maximality yields the following useful characterization of $\mathscr{A}$, where $\mathscr{A}_0 = \mathscr{A} \cap \ker\Phi$ (see [1]):

$$\mathscr{A} = \{x \in \mathscr{M} : \tau(xa) = 0, \forall a \in \mathscr{A}_0\}. \tag{1}$$

Given $0 < p \leq \infty$ we denote by $L_p(\mathscr{M})$ the usual noncommutative L_p-spaces associated wit $(\mathscr{M}, \tau)$. Recall that $L_\infty(\mathscr{M}) = \mathscr{M}$, equipped with the operator norm (see [22, 4]). The norm of $L_p(\mathscr{M})$ will be denoted by $\|\cdot\|_p$ (see [14, 15].) For $p < \infty$ we define $H_p(\mathscr{A})$ to be closure of $\mathscr{A}$ in $L_p(\mathscr{M})$, and for $p = \infty$ we simply set $H_\infty(\mathscr{A}) = \mathscr{A}$ for convenience. These are so called Hardy spaces associated with $\mathscr{A}$. They are noncommutative extensions of the classical Hardy space on the torus $\mathscr{T}$. We refer to [1, 20, 22] for more examples. The theory of vector-valued noncommutative L_p-spaces are introduced by Pisier in [21]. Pisier considered the case $\mathscr{M}$ is hyperfinite. We refer the reader notably to the recent work by Defant/Junge [9]. Junge and Xu introduced the spaces $L_p^{(r,s)}(\mathscr{M}, \ell_\infty)$ and $L_p(\mathscr{M};\ell_1)$ (see also [17]). They proved that both spaces $L_p^{(r,s)}(\mathscr{M}, \ell_\infty)$ and $L_p(\mathscr{M};\ell_1)$ are Banach spaces for $1 \leq p \leq \infty$ and basic properties of these spaces. We refer the reader notably to the recent work by Defant/Junge [9]. We now define the analogue of $L_p^{(r,s)}(\mathscr{M};\ell_\infty)$ and $L_p(\mathscr{M};\ell_1)$ spaces by a similar way (see [3, 24]).

Definition 1. *Let $1 \leq p < \infty$ and $1 \leq r, s \leq \infty$ such that $1/p = 1/r + 1/s$.*

(i) We define $H_p^{(r,s)}(\mathscr{A}, \ell_\infty)$ as the space of all sequences $x = (x_n)_{n\geq 1}$ in $H_p(\mathscr{A})$ which admit a factorization of the following form: there are $a \in H_r(\mathscr{A})$, $b \in H_s(\mathscr{A})$ and a bounded sequence $y = (y_n) \subset \mathscr{A}$ such that

$$x_n = a y_n b, \forall n \geq 1.$$

Given $x \in H_p^{(r,s)}(\mathscr{A}, \ell_\infty)$ define

$$\|x\|_{p;(r,s)} = \inf\{\|a\|_r \sup_n \|y_n\|_\infty \|b\|_s\},$$

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where the infimum runs over all factorizations of (x_n) *as above. The spaces*

$$H_p^{right}(\mathscr{A};\ell_\infty) := H_p^{(\infty,p)}(\mathscr{A};\ell_\infty)$$

and

$$H_p^{left}(\mathscr{A};\ell_\infty) := H_p^{(p,\infty)}(\mathscr{A};\ell_\infty)$$

will be of special interest - all sequences (x_n) *which allow uniform factorizations* $x_n = y_n b$ *and* $x_n = a y_n$ *with* $a, b \in H_p(\mathscr{A})$ *and a bounded sequence* $(y_n) \subset \mathscr{A}$*, respectively. Moreover, in the symmetric case put*

$$H_p(\mathscr{A};\ell_\infty) := H_p^{(2p,2p)}(\mathscr{A};\ell_\infty).$$

(ii) Let $1 \leq p \leq \infty$*. We define* $H_p(\mathscr{A};\ell_1)$ *as the space of all sequences* $x = (x_n)_{n\geq 1}$ *in* $H_p(\mathscr{A})$ *which can be decomposed as*

$$x_n = \sum_{k=1}^{\infty} u_{kn} v_{nk}, \forall n \geq 1$$

for two families $(u_{kn})_{k,n\geq 1}$ *and* $(v_{nk})_{n,k\geq 1}$ *in* $H_{2p}(\mathscr{A})$ *such that*

$$\sum_{k,n=1}^{\infty} u_{kn} u_{kn}^* \in L_p(\mathscr{M}) \text{ and } \sum_{n,k=1}^{\infty} v_{nk}^* v_{nk} \in L_p(\mathscr{M}).$$

In this space we define the norm

$$\|x\|_{H_p(\mathscr{A};\ell_1)} = \inf\{\|\sum_{k,n=1}^{\infty} u_{kn} u_{kn}^*\|_p^{1/2} \|\sum_{n,k=1}^{\infty} v_{nk}^* v_{nk}\|_p^{1/2}\},$$

where the infimum runs over all decompositions of x as above.

Thus $H_p(\mathscr{A}) = [\mathscr{A}]_p$. Formula (1) admits the following $H_p(\mathscr{A})$ analogue proved by Saito [23]:

$$H_p(\mathscr{A}) = \{x \in L_p(\mathscr{M}) : \tau(xa) = 0, \forall a \in \mathscr{A}_0\}, \ 1 \leq p < \infty \tag{2}$$

Then in [2] Bekjan and Xu proved that formula (2) holds for every $0 < p < q \leq \infty$. This noncommutative Hardy spaces have received a lot of attention since Arveson's pioneer work. We refer the reader a series of newly finished papers by Blecher/Labuschagne [5–7], whereas more references on previous works can be found in the survey paper [22]. Most results on the classical Hardy spaces on the torus have been established in this noncommutative setting. Here we mention some of them directly related with the objective of this paper. One of them is the Saito's theorem. The main purpose of the present paper is to extend all these results to the spaces above we define.

MAIN RESULTS

To gain a very first understanding on $H_p(\mathscr{A};\ell_1)$ and $H_p(\mathscr{A};\ell_\infty)$ spaces above we define, we need the following propositions.

Proposition 1. *Let* $1 \leq p \leq \infty$*. Then for any* $x \in H_p(\mathscr{A};\ell_1)$ *we have*

$$\left\|\sum_{n=1}^{\infty} x_n\right\|_p \leq \|x\|_{H_p(\mathscr{A};\ell_1)}.$$

If in addition x is positive,

$$\left\|\sum_{n=1}^{\infty} x_n\right\|_p = \|x\|_{H_p(\mathscr{A};\ell_1)}.$$

This means that, a positive sequence $x = (x_n)$ *(i.e.* $x_n \geq 0$ *for all n) belongs to* $H_p(\mathscr{A};\ell_1)$ *if and only if*

$$\sum_{n=1}^{\infty} x_n \in H_p(\mathscr{A}).$$

Proposition 2. *Let $1 \leq p \leq \infty$. A positive sequence $x = (x_n)$ (i.e. $x_n \geq 0$ for all n) belongs to $H_p(\mathscr{A};\ell_\infty)$ if and only if there exists a positive $a \in H_p(\mathscr{A})$ such that*

$$x_n \leq a \quad \forall n \geq 1.$$

Proof. Let $(x_n) \in H_p(\mathscr{A})$. Assume that there exists positive $a \in H_p(\mathscr{A})$ such that $x_n \leq a$ $\forall n \geq 1$. Then, by Remark 2.3 in [9], there exists a contraction operator $u_n \in \mathscr{M}$ such that $x_n^{1/2} = u_n a^{1/2}$, so $x_n^{1/2} = a^{1/2} u_n^* u_n a^{1/2}$. Thus $x \in L_p(\mathscr{M};\ell_\infty)$ and $\|x\|_{L_p(\mathscr{M};\ell_\infty)} \leq \|a\|_p$ (see [17]). Then by using Proposition 2.1 in [3] , we obtain $x \in H_p(\mathscr{A};\ell_\infty)$. On the other hand if $x \in H_p(\mathscr{A};\ell_\infty)$ is positive, then for all $n \geq 1$ we can find a positive $a \in H_p(\mathscr{A})$ and positive contractions $y_n \in \mathscr{A}$ such that $x_n = a^{1/2} y_n a^{1/2}$. From this it is easy to show that $x_n \leq a$, which is the conclusion. □

Theorem 1. *Let $2 \leq r,s \leq \infty$ such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$ and let $1 \leq q \leq \infty$. Then $H_p^{(r,s)}(\mathscr{A},\ell_\infty)$ and $H_q(\mathscr{A};\ell_1)$ are Banach spaces, respectively.*

Proof. First we prove the part on $H_q(\mathscr{A};\ell_1)$. Let $x^{(i)} \in H_q(\mathscr{A};\ell_1)$ with $i = 1,2$ and $\varepsilon > 0$. Choose $(u_{kn}^{(i)})$ and $(v_{nk}^{(i)})$ in $H_{2q}(\mathscr{A})$ such that

$$x_n^{(i)} = \sum_{k=1}^{\infty} u_{kn}^{(i)} v_{nk}^{(i)} \ \forall n$$

and

$$\left\| \sum_{k,n=1}^{\infty} u_{kn}^{(i)} (u_{kn}^{(i)})^* \right\|_q = \left\| \sum_{n,k=1}^{\infty} (v_{nk}^{(i)})^* v_{nk}^{(i)} \right\|_q \leq \|x^{(i)}\|_{H_q(A;\ell_1)} + \varepsilon.$$

Then

$$x^{(1)} + x^{(2)} = \sum_{i=1}^{2} \sum_{k=1}^{\infty} u_{kn}^{(i)} v_{nk}^{(i)}$$

is a decomposition of $x^{(1)} + x^{(2)}$ and

$$\left\| \sum_{i=1}^{2} \sum_{k,n=1}^{\infty} u_{kn}^{(i)} (u_{kn}^{(i)})^* \right\|_q \leq \|x^{(1)}\|_{H_q(\mathscr{A};\ell_1)} + \|x^{(2)}\|_{H_q(\mathscr{A};\ell_1)} + 2\varepsilon.$$

A similar inequality holds for $v_{nk}^{(i)}$. It then follows that

$$\begin{aligned} \|x^{(1)} + x^{(2)}\|_{H_q(\mathscr{A};\ell_1)} &\leq \left\| \sum_{i=1}^{2} \sum_{k,n=1}^{\infty} u_{kn}^{(i)} (u_{kn}^{(i)})^* \right\|_q^{1/2} \left\| \sum_{i=1}^{2} \sum_{n,k=1}^{\infty} (v_{nk}^{(i)})^* v_{nk}^{(i)} \right\|_q^{1/2} \\ &\leq \|x^{(1)}\|_{H_q(\mathscr{A};\ell_1)} + \|x^{(2)}\|_{H_q(\mathscr{A};\ell_1)} + 2\varepsilon. \end{aligned}$$

Therefore, $\|\cdot\|_{H_q(A;\ell_1)}$ verifies triangle inequality. On the other hand it is trivial $\|x_n\|_q \leq \|x\|_{H_q(\mathscr{A};\ell_1)}$ $\forall n \geq 1$. It follows that $\|\cdot\|_{H_q(\mathscr{A};\ell_1)}$ is a norm. To prove its completeness, it suffices to show that if $\sum_{i=1}^{\infty} \|x^{(i)}\|_{H_q(\mathscr{A};\ell_1)} < \infty$, then the series $\sum_{i=1}^{\infty} x^{(i)}$ converges in $H_q(\mathscr{A};\ell_1)$. This is proved by an argument similar to the previous one.

We turn to $H_p^{(r,s)}(\mathscr{A};\ell_\infty)$. Let us first check that $\|\cdot\|_{p;(r,s)}$ satisfies triangle inequality provided $r,s \geq 2$. Let $(h_n^{(1)}), (h_n^{(2)}) \in H_p^{(r,s)}(\mathscr{A},\ell_\infty)$, choose a factorization of $h^{(j)}$ with $j = 1,2$:

$$h_n^{(j)} = a^{(j)} x_n^{(j)} b^{(j)} \ \forall n$$

such that

$$\|a^{(j)}\|_r = \|b^{(j)}\|_s = \|(h_n^{(j)})\|_{p;(r,s)}^{\frac{1}{2}}$$

and

$$\sup_n \|x_n^{(j)}\|_\infty \leq 1+\varepsilon, \ \ where \ \ j=1,2.$$

Indeed, for any $\varepsilon > 0$ choose a factorization $h_n^{(j)} = c^{(j)} y_n^{(j)} d^{(j)}$ $\forall n \geq 1$ with $j=1,2$ such that

$$c^{(j)} \in H_r(\mathscr{A}), \ \ d^{(j)} \in H_s(\mathscr{A}), \ \ \sup_n \|y_n^{(j)}\|_\infty = \alpha$$

and

$$\|h_n^{(j)}\|_{p;(r,s)}(1+\varepsilon) \geq \|\alpha^{\frac{1}{2}} c^{(j)}\|_r \sup_n \|\frac{y_n^{(j)}}{\alpha}\|_\infty \|\alpha^{\frac{1}{2}} d^{(j)}\|_s.$$

Then by choosing

$$a^{(j)} = \frac{\alpha\|(h_n^{(j)})\|_{p;(r,s)}^{1/2} c^{(j)}}{\|\alpha^{\frac{1}{2}} c^{(j)}\|_r}, \ \ b^{(j)} = \frac{\alpha\|(h_n^{(j)})\|_{p;(r,s)}^{1/2} d^{(j)}}{\|\alpha^{\frac{1}{2}} d^{(j)}\|_s}$$

and

$$x_n^{(j)} = \frac{\|\alpha^{\frac{1}{2}} c^{(j)}\|_r \|\alpha^{\frac{1}{2}} d^{(j)}\|_s y_n^{(j)}}{\alpha\|(h_n^{(j)})\|_{p;(r,s)}},$$

we obtain

$$a^{(j)} x_n^{(j)} b^{(j)} = c^{(j)} y_n^{(j)} d^{(j)} = h_n^{(j)}, \ \ j=1,2$$

and

$$\|a^{(j)}\|_r = \|(h_n^{(j)}\|_{p;(r,s)}^{1/2},$$

$$\|b^{(j)}\|_s = \|(h_n^{(j)}\|_{p;(r,s)}^{1/2}.$$

Let $a^{(j)} = |(a^{(j)})^*| u^{(j)}$ and $b^{(j)} = v^{(j)}|b^{(j)}|$ be the polar decompositions of $(a^{(j)})^*$ and $b^{(j)}$, respectively. Then substituting $x_n^{(j)}$ by $u^{(j)} x_n^{(j)} v^{(j)}$, we may assume that the $a^{(j)}$'s and $b^{(j)}$'s are positive. Define operators: $a := (|(a^{(1)})^*|^2 + |(a^{(2)})^*|^2 + \varepsilon)^{\frac{1}{2}}$ and $b := (|b^{(1)}|^2 + |b^{(2)}|^2 + \varepsilon)^{\frac{1}{2}}$; clearly,

$$\|a\|_r \leq (\|(a^{(1)})^*\|_r^2 + \|(a^{(2)})^*\|_r^2 + \varepsilon)^{\frac{1}{2}} = (\|(h_n^{(1)})\|_{p;(r,s)} + \|(h_n^{(2)})\|_{p;(r,s)} + \varepsilon)^{\frac{1}{2}},$$

a similar inequality holds for b with norm $\|\cdot\|_s$. By Remark in [9] there exist contractions $\omega^{(j)}, \theta^{(j)} \in \mathscr{M}$ such that $|(a^{(j)})^*| = a(\omega^{(j)})^*$, $|b^{(j)}| = \theta^{(j)} b$ and

$$(\omega^{(1)})^*\omega^{(1)} + (\omega^{(2)})^*\omega^{(2)} = r(a^2), \ (\theta^{(1)})^*\theta^{(1)} + (\theta^{(2)})^*\theta^{(2)} = r(b^2).$$

And, since $a^{-1}, b^{-1} \in \mathscr{M}$ and $(a^{-1})^{-1} = a, b \in L_r(\mathscr{M})$, by Theorem 3.1. in [2] there exist the unitary operators $\nu^{(1)}, \nu^{(2)} \in \mathscr{M}$ and $w^{(1)}, w^{(2)} \in \mathscr{A}$ such that $a^{-1} = \nu^{(1)} w^{(1)}$ and $b^{-1} = w^{(2)} \nu^{(2)}$, where $(w^{(1)})^{-1}, (w^{(2)})^{-1} \in H_r$. Obviously,

$$\begin{aligned} h_n^{(1)} + h_n^{(2)} = (w^{(1)})^{-1}[(\nu^{(1)})^{-1}(\omega^{(1)})^* u^{(1)} x_n^{(1)} v^{(1)} \theta^{(1)} \\ + (\omega^{(2)})^* u^{(2)} x_n^{(2)} v^{(2)} \theta^{(2)} (\nu^{(2)})^{-1}](w^{(2)})^{-1}. \end{aligned}$$

Define the sequence

$$y_{(\cdot)} := (\nu^{(1)})^{-1}(\omega^{(1)})^* u^{(1)} x^{(1)}_{(\cdot)} v^{(1)} \theta^{(1)} + (\omega^{(2)})^* u^{(2)} x^{(2)}_{(\cdot)} v^{(2)} \theta^{(2)} (\nu^{(2)})^{-1}.$$

Since $y_n = (w^{(1)})^{-1}[h_n^{(1)} + h_n^{(2)}](w^{(2)})^{-1} \in H_p(\mathscr{A})$ by Proposition 3.3. in [2] $y_n \in H_p(\mathscr{A}) \cap \mathscr{M} = \mathscr{A}$. Consider for each fixed n the following mapping:

$$U : \mathbf{M}_2(\mathscr{M}) \to \mathbf{M}_2(\mathscr{M})$$

defined by

$$U(X) = \begin{pmatrix} (\omega^{(1)})^* & (\omega^{(2)})^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^{(1)} & 0 \\ 0 & u^{(2)} \end{pmatrix} X \begin{pmatrix} v^{(1)} & 0 \\ 0 & v^{(2)} \end{pmatrix} \begin{pmatrix} \theta^{(1)} & 0 \\ \theta^{(2)} & 0 \end{pmatrix},$$

where

$$X = \begin{pmatrix} y_n^{(1)} & 0 \\ 0 & y_n^{(2)} \end{pmatrix} \in \mathbf{M}_2(\mathscr{M}).$$

We need to show that $\|y_n\| \leq 1$.

Indeed,

$$\begin{aligned} \|y_n\| &= \|(\nu^{(1)})^{-1}[(\omega^{(1)})^* u^{(1)} x_n^{(1)} v^{(1)} \theta^{(1)} + (\omega^{(2)})^* u^{(2)} x_n^{(2)} v^{(2)} \theta^{(2)}](\nu^{(2)})^{-1}\| \\ &= \|(\omega^{(1)})^* u^{(1)} x_n^{(1)} v^{(1)} \theta^{(1)} + (\omega^{(2)})^* u^{(2)} x_n^{(2)} v^{(2)} \theta^{(2)}\| = \|U(X)\| \\ &\leq \left\| \begin{pmatrix} (\omega^{(1)})^* & (\omega^{(2)})^* \\ 0 & 0 \end{pmatrix} \right\| \left\| \begin{pmatrix} u^{(1)} & 0 \\ 0 & u^{(2)} \end{pmatrix} \right\| \|X\| \left\| \begin{pmatrix} v^{(1)} & 0 \\ 0 & v^{(2)} \end{pmatrix} \right\| \left\| \begin{pmatrix} \theta^{(1)} & 0 \\ \theta^{(2)} & 0 \end{pmatrix} \right\| \\ &\leq \|\omega_1^* \omega_1 + \omega_2^* \omega_2\|^{\frac{1}{2}} \|\theta_1^* \theta_1 + \theta_2^* \theta_2\|^{\frac{1}{2}} \leq \|r(a^2)\|^{\frac{1}{2}} \|r(b^2)\|^{\frac{1}{2}} \leq 1. \end{aligned}$$

So,

$$\begin{aligned} \|(h_n^{(1)} + h_n^{(2)})\|_{H_p^{(r,s)}(\mathscr{A},\ell_\infty)} &\leq \|c\|_r \sup_n \|y_n\| \|d\|_s \leq (\| |(c^{(1)})^*|^2 \|_{\frac{r}{2}} + \| |(c^{(2)})^*|^2 \|_{\frac{r}{2}} + \varepsilon)^{\frac{1}{2}} \\ &\times (\| |d^{(1)}|^2 \|_{\frac{r}{2}} + \| |d^{(2)}|^2 \|_{\frac{r}{2}} + \varepsilon)^{\frac{1}{2}} \leq (\|c^{(1)}\|_r^2 \\ &+ \|c^{(2)}\|_r^2 + \varepsilon)^{\frac{1}{2}} (\|d^{(1)}\|_r^2 + \|d^{(2)}\|_r^2 + \varepsilon)^{\frac{1}{2}} \\ &= \|(h_n^{(1)})\|_{H_p^{(r,s)}(\mathscr{A},\ell_\infty)} + \|(h_n^{(2)})\|_{H_p^{(r,s)}(\mathscr{A},\ell_\infty)} + \varepsilon. \end{aligned}$$

Then letting $\varepsilon \to 0$ we obtain the desired triangle inequality. To show the completeness, it suffices to show that if $\sum_{i=1}^\infty \|x^{(i)}\|_{H_p^{(r,s)}(\mathscr{A};\ell_\infty)} < \infty$, then the series $\sum_{i=1}^\infty x^{(i)}$ converges in $H_p^{(r,s)}(\mathscr{A};\ell_\infty)$. This is proved by an argument similar to the previous one. □

Proposition 3. *Let $1 \leq p < \infty$. Then we have the following, where $H_p^0(\mathscr{A};\ell_\infty) = \{x \in H_p(\mathscr{A};\ell_\infty) : \Phi(x_n) = 0, \ \forall n\}$:*

$$H_p(\mathscr{A};\ell_\infty) = \{x \in L_p(\mathscr{M};\ell_\infty) : \tau(x_n c) = 0, \ for \ all \ c \in \mathscr{A}_0 \ and \ n\}. \tag{3}$$

Moreover,

$$H_p^0(\mathscr{A};\ell_\infty) = \{x \in L_p(\mathscr{M};\ell_\infty) : \tau(x_n c) = 0, \ for \ all \ c \in \mathscr{A} \ and \ n\}. \tag{4}$$

Proof. The inclusion $H_p(\mathscr{A};\ell_\infty) \subset \{x \in L_p(\mathscr{M};\ell_\infty) : \tau(x_n c) = 0, \ for \ all \ c \in \mathscr{A}_0 \ and \ n\}$ is clearly. Let $y \in \{x \in L_p(\mathscr{M};\ell_\infty) : \tau(x_n c) = 0, \ for \ all \ c \in \mathscr{A}_0 \ and \ n\}$. Then by Lemma 2.1 (i) in [3] there exist $a \in H_{2p}(\mathscr{A})$, $b \in H_{2p}(\mathscr{A})$ and $z_n \in \mathscr{M}$ such that

$$y_n = a z_n b \ \forall n,$$

where $a^{-1}, b^{-1} \in \mathscr{A}$ and $\sup_n \|y_n\|_\infty \leq 1$. On the other hand we have $\tau(y_n c) = 0, \ \forall c \in \mathscr{A}_0$. Since $a^{-1} s b^{-1} \in \mathscr{A}_0, \ \forall s \in \mathscr{A}_0$, substituting c by $a^{-1} s b^{-1}$ we obtain $z_n \in \mathscr{A}$ (see also [23]), so $(y_n) \in H_p(\mathscr{A};\ell_\infty)$. Similarly we can prove (4), which is the conclusion. □

Remark 1. *The previous proposition is also holds for the $H_p(\mathscr{A};\ell_1)$ space with similar proof.*

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