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Chameleon stars

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We consider a gravitating spherically symmetric configuration consisting of a scalar field nonminimally coupled to ordinary matter in the form of a perfect fluid. For this system we find static, regular, asymptotically flat solutions for both relativistic and nonrelativistic cases. It is shown that the presence of the nonminimal interaction leads to substantial changes both in the radial matter distribution of the star and in the star's total mass. A simple stability test indicates that, for the choice of parameters used in the paper, the solutions are unstable.

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I. INTRODUCTION

In cosmology, scalar fields play a key role in models of both the inflation era of the early Universe [1] and the current accelerated expansion [2]. Scalar fields are also widely used to study smaller scale objects such as the effect of a dark matter, scalar field on the structure of galaxies [3]. Moving to still smaller scales, there have been studies of boson stars [4]—compact (usually spherically symmetric) configurations formed from gravitating scalar field(s). The sizes of such configurations varies from the microscopic ("gravitational atom") up to sizes corresponding to massive black holes such as those presumed to exist at the center of many galaxies including our own. In this connection, it is quite natural to investigate the role that such scalar fields may have in the processes of formation and evolution of galaxies and more compact astrophysical structure—stars and their clusters. As applied to the stars, it is possible to imagine a situation where scalar fields can exist inside usual stars consisting of ordinary matter. The existence of such scalar fields would undoubtedly have an impact on the star's inner structure. One such example is the model of a star consisting of both ordinary polytropic matter and a ghost scalar field [5]. The presence of the field leads to the appearance of a tunnel at the center of such stellar configurations which provides new geometric and physical properties to such objects. Another example of such stellar objects involves fermion fields interacting with a real scalar field Φ not only gravitationally but also through the Yukawa coupling of the type $-f\bar{\psi}\psi\Phi$, where f is a Yukawa type coupling constant [6]. Further examples of mixed configurations consisting of both boson and fermion fields can be found in Refs. [7–9].

The investigation in the present work is along the direction of the above works—we consider a scalar field interacting nonminimally with ordinary matter in the form of a perfect polytropic fluid. An interaction similar to this was used to describe the evolution of dark energy within the framework of chameleon cosmologies [10,11]. In chameleon cosmologies, the properties of the scalar (chameleon) field depend strongly on the environment [12] into which it is embedded. For example, the mass of the scalar field can change depending on the background environment [12]. As will be shown below, the presence of such an interaction between the polytropic fluid and the scalar field results in a substantial change of the inner structure of the polytropic star. This change is related to the fact that the behavior of the scalar field depends the "environment" (i.e. the fluid) into which it is embedded, as is the case with chameleon cosmology models. Thus, we call these configurations *chameleon stars*.

In building our model of a chameleon star, we will use a real scalar field. In the bulk of the literature complex (charged) scalar fields are more often used in constructing models of boson stars. The charge of the complex scalar field generically leads to repulsion which counteracts the attraction of gravity and thus gives a physical basis for the existence of these boson star solutions. Real scalar fields have received less attention because they do not carry a charge, and constructing regular stable solutions is the more difficult task in this case. The known static solutions for real scalar fields are either singular [13,14], or use phantom scalar fields [15–17]. Singular solutions with trivial topology and nonphantom fields have been found for both massless fields [13] and for self-interacting fields [14]. The latter ordinary scalar field solutions are asymptotically flat but the issue of their stability is not quite clear (for further discussion see Ref. [18]). In [19] regular, static solutions in the presence of the cosmological constant were found. In this case, the space-time asymptotically approaches de Sitter space-time. The linear stability analysis of these solutions indicate that they are unstable. In contrast, the paper [20] found regular, static solutions which asymptotically

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approached anti-de Sitter space-time, which were stable under a linear stability analysis. In the present paper, the real scalar field nonminimally interacts with the polytropic fluid. As will be shown below, this leads to the existence of regular, asymptotically flat solutions.

In the *absence* of gravity, Derrick's theorem [21,22] forbids the existence of nontrivial, stable, regular static $D \geq 3$ dimensional solutions from only scalar fields. This theorem assumes that the potential energy of the system is some non-negative function, $V(\varphi_i) > 0$, vanishing only at its absolute minima. A partial extension of Derrick's theorem to the case of general relativity has been studied in [23,24] where it was shown that the existence of asymptotically flat particlelike solutions with a regular center and normal (nonphantom) scalar fields is also impossible if $V(\varphi) > 0$. The distinction between the nongravitational form of Derrick's theorem [21,22] and the gravitational variant [23,24] is that the latter does not touch on the stability of the solutions.

If one allows other fields besides just scalar fields (e.g. vector or fermion fields) then there are static solutions when the spatial dimensionality is three or more. In the absence of gravity, one has the 't Hooft-Polyakov monopole which is a finite energy, stable solution for Yang-Mills vector fields plus scalar fields [25,26]. In the presence of gravity plus a real scalar field, there are boson star solutions (for a review see [4]) which however have one or more of the bad features listed above (e.g. nontrivial topology, not asymptotically flat, not stable).

In this paper, we study the gravitating system of a real scalar field plus a perfect fluid and show that for this system there are regular, three-dimensional, static solutions. The most important feature of this system, aside from the gravitational interaction, is the presence of the nonminimal coupling between the scalar field and the perfect fluid. It is this interaction which allows us to avoid the restrictions of the gravitational version of Derrick's theorem [23,24], and find regular static solutions even if the potential energy of the scalar field satisfies $V(\varphi) > 0$. We take a polytropic equation of state for the perfect fluid. Stellar models using polytropic fluids were investigated in detail both within the framework of Newtonian gravitational theory [27] and for strong gravitational fields [28]. Both of these studies hinted at the existence of finite size, regular solutions. These solutions were successfully used to give a description of both nonrelativistic stars and stars where relativistic effects were important, e.g. neutron stars. As will be shown below, the inclusion of the nonminimal interaction between the polytropic fluid and the scalar field leads to regular solutions which, however, are generally different from usual polytropic stars.

The paper is organized as follows: In Sec. II, the general equations describing a static configuration consisting of a real scalar field coupled to a perfect polytropic fluid are derived. In Sec. III, these equations are written for a particular case when the potential energy of the scalar field is chosen to have a quadratic mass term and a quartic self-interaction term. For this potential energy, in Sec. IV we give the results of the numerical calculations for this potential for both relativistic and nonrelativistic cases, and discuss the issue of the stability of the solutions obtained. Next, in Sec. V we present a simple analytical solution for the nonrelativistic case with the scalar field taken to be massless and for a special choice of the coupling function. Finally, in Sec. VI we summarize the main results and give some speculations about the physical applications of these chameleon star configurations to neutron stars and living stars (i.e. stars still on the main sequence).

II. DERIVATION OF EQUATIONS FOR A STATIC CONFIGURATION

As discussed in the introduction, we consider a gravitating system of a real scalar field coupled to a perfect fluid. The Lagrangian for this system is

$$L = -\frac{R}{16\pi G} + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) + f(\varphi)L_m. \tag{1}$$

Here, φ is the real scalar field with the potential $V(\varphi)$; L_m is the Lagrangian of the perfect isotropic fluid i.e. a fluid with only one radial pressure; $f(\varphi)$ is some function describing the nonminimal interaction between the fluid and the scalar field. The case $f = 1$ corresponds to the absence of the nonminimal coupling, but even in this case the two sources are still coupled via gravity.

We choose the Lagrangian for the isentropic perfect fluid to have the form $L_m = p$ [29,30]. Using this Lagrangian, the corresponding energy-momentum tensor is (details are given in the Appendix)

$$T_i^k = f[(\rho + p)u_i u^k - \delta_i^k p] + \partial_i\varphi\partial^k\varphi - \delta_i^k\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right], \tag{2}$$

where ρ and p are the density and the pressure of the fluid, u^i is the four-velocity (here and throughout the paper we set $c = 1$). We take the static metric of the form

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2 d\Omega^2, \tag{3}$$

where $d\Omega^2$ is the metric on the unit 2-sphere. The

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

components of the Einstein equations for the metric (3) and the energy-momentum tensor (2) are

$$G_0^0 = -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} = 8\pi G T_0^0 = 8\pi G\left[f\rho + \frac{1}{2}e^{-\lambda}\varphi'^2 + V(\varphi)\right], \tag{4}$$

$$G_1^1 = -e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) + \frac{1}{r^2} = 8\pi G T_1^1$$
$$= 8\pi G\left[-fp - \frac{1}{2}e^{-\lambda}\varphi'^2 + V(\varphi)\right]. \quad (5)$$

The equation for the scalar field φ coming from the Lagrangian (1) is

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left[\sqrt{-g}g^{ik}\frac{\partial\varphi}{\partial x^k}\right] = -\frac{dV}{d\varphi} + L_m\frac{df}{d\varphi}.$$

Using this field equation above with the perfect fluid $L_m = p$, and the metric (3), gives the following scalar field equation

$$\varphi'' + \left[\frac{2}{r} + \frac{1}{2}(\nu' - \lambda')\right]\varphi' = e^{\lambda}\left(\frac{dV}{d\varphi} - p\frac{df}{d\varphi}\right), \quad (6)$$

where the prime denotes differentiation with respect to r. The Einstein field equations are not all independent because of the relation $T^k_{i;k} = 0$. The $i = 1$ component of this equation has the form

$$\frac{\partial T_1^1}{\partial r} + \frac{1}{2}(T_1^1 - T_0^0)\nu' + \frac{2}{r}\left[T_1^1 - \frac{1}{2}(T_2^2 + T_3^3)\right] = 0. \quad (7)$$

Taking into account the expressions

$$T_2^2 = T_3^3 = -fp + \frac{1}{2}e^{-\lambda}\varphi'^2 + V(\varphi),$$

and T_0^0, T_1^1 from (4) and (5) and using (6), allows us to write (7) as

$$\frac{dp}{dr} = -\frac{1}{2}(\rho + p)\frac{d\nu}{dr}. \quad (8)$$

The matter Lagrangian used here, $L_m = p$, is not the only possibility. Other variants can be found in [31]. However, the choice $L_m = p$ has the simplifying feature (8) does not contain additional terms involving the coupling function $f(\varphi)$.

For a polytropic equation of state, one has

$$p = k\rho^{\gamma}, \quad (9)$$

where k, γ are constants. Now one can introduce the new variable θ defined as [32]

$$\rho = \rho_c\theta^n. \quad (10)$$

Here, ρ_c is the central density, and the constant n, the polytropic index, is related to γ via $n = 1/(\gamma - 1)$. Putting these definitions together gives Eq. (9) in the form

$$p = k\rho^{\gamma} = k\rho^{1+1/n} = k\rho_c^{1+1/n}\theta^{n+1}. \quad (11)$$

Using (11) in Eq. (8) leads to

$$2\sigma(n+1)\frac{d\theta}{dr} = -(1 + \sigma\theta)\frac{d\nu}{dr}, \quad (12)$$

with $\sigma = k\rho_c^{1/n} = p_c/\rho_c$ and p_c is the pressure of the fluid at the center of the configuration. This equation may be integrated to give e^{ν} in terms of θ:

$$e^{\nu} = e^{\nu_c}\left(\frac{1 + \sigma}{1 + \sigma\theta}\right)^{2(n+1)}, \quad (13)$$

where e^{ν_c} is the value of e^{ν} at the center of the configuration where $\theta = 1$. The integration constant ν_c, corresponds to the value of ν at the center of the configuration. It is determined by requiring $e^{\nu} = 1$ at infinity that the spacetime is asymptotically flat.

The gravitating system of a real scalar field interacting with a perfect fluid is characterized by three unknown functions—λ, θ and φ. These three functions are determined by the three Eqs. (4)–(6), and also by the relation (13). We now rewrite these equations by the introduction of a new function $u(r)$ [28]

$$u(r) = \frac{r}{2GM}(1 - e^{-\lambda}) \rightarrow e^{-\lambda} = 1 - \frac{2GMu}{r}. \quad (14)$$

Here, M is the mass of the configuration within the range $0 \le r \le r_b$, where r_b is the boundary of the fluid where $\theta = 0$. Using this function, Eq. (4) becomes

$$M\frac{du}{dr} = 4\pi r^2\left[f\rho + \frac{1}{2}\left(1 - \frac{2GMu}{r}\right)\varphi'^2 + V\right]. \quad (15)$$

From (14), one can define $M(r) \equiv Mu(r)$ which can be interpreted as the total mass of the configuration in the range $[0, r]$. This mass has contributions from both the fluid and the scalar field, within a sphere of coordinate radius r. To avoid a singularity in $M(r)$ at the origin, one has to put $u(0) = 0$ [28]. This corresponds to the fact that the mass at the origin is equal to zero, i.e. $M(0) = 0$.

In anticipation of analyzing the system of Eqs. (4)–(6)—numerically, we introduce the following dimensionless variables:

$$\xi = Ar, \quad \upsilon(\xi) = \frac{A^3M}{4\pi\rho_c}u(r), \quad \phi(\xi) = \left[\frac{4\pi G}{\sigma(n+1)}\right]^{1/2}\varphi(r),$$
$$\text{where } A = \left[\frac{4\pi G\rho_c}{(n+1)k\rho_c^{1/n}}\right]^{1/2}, \quad (16)$$

A has the dimensions of an inverse length. With this, one can rewrite Eqs. (4) and (5) in the form

$$\frac{d\upsilon}{d\xi} = \xi^2\left\{f\theta^n + \frac{1}{2}\left[1 - 2\sigma(n+1)\frac{\upsilon}{\xi}\right]\left(\frac{d\phi}{d\xi}\right)^2 + \tilde{V}\right\}, \quad (17)$$

$$\xi^2\frac{1 - \frac{2\sigma(n+1)\upsilon}{\xi}}{1 + \sigma\theta}\frac{d\theta}{d\xi} = \xi^3\left[f\theta^n(1 - \sigma\theta) + 2\tilde{V} - \frac{1}{\xi^2}\frac{d\upsilon}{d\xi}\right] - \upsilon, \quad (18)$$

where $\tilde{V} = V/\rho_c$ is the dimensionless potential energy of the field.

Next, using (12), one can rewrite Eq. (6) as follows:

$$\frac{d^2\phi}{d\xi^2} + \left\{\frac{2}{\xi} - \frac{\sigma(n+1)}{1+\sigma\theta}\left[\frac{d\theta}{d\xi} + \frac{1+\sigma\theta}{1-\frac{2\sigma(n+1)v}{\xi}}\frac{1}{\xi}\left(\frac{dv}{d\xi} - \frac{v}{\xi}\right)\right]\right\}\frac{d\phi}{d\xi} = \left[1 - 2\sigma(n+1)\frac{v}{\xi}\right]^{-1}\left(\frac{d\tilde{V}}{d\phi} - \sigma\theta^{n+1}\frac{df}{d\phi}\right). \tag{19}$$

Thus, the static configuration under consideration is described by the three Eqs. (17)–(19).

III. CONFIGURATION WITH A MASS AND A QUARTIC SELF-INTERACTION TERM

In this section, we will show that there are nonsingular, finite-mass solutions of Eqs. (17)–(19). First, we specify the boundary conditions. Using the above dimensionless variables, we "normalize" θ to unity at the center of the configuration $\xi = 0$

$$\theta_0 \equiv \theta(0) = 1. \tag{20}$$

From Eq. (17), one can show that $v \to 0$ like ξ^3 as $\xi \to 0$. Combining this with Eq. (18), one in turn finds $d\theta/d\xi \to 0$ as $\xi \to 0$. Bearing in mind that we are looking for regular solutions, we define the boundary conditions in the vicinity of $\xi \approx 0$ as

$$\theta \approx \theta_0 + \frac{\theta_2}{2}\xi^2, \quad v \approx v_3\xi^3, \quad \phi \approx \phi_0 + \frac{\phi_2}{2}\xi^2, \tag{21}$$

where ϕ_0 corresponds to the initial value of the scalar field ϕ, the parameters θ_2, v_3 are arbitrary, and the value of the coefficient ϕ_2 is defined from Eq. (19) as

$$\phi_2 = \frac{1}{3}\left[\left(\frac{d\tilde{V}}{d\phi}\right)_0 - \sigma\theta_0^{n+1}\left(\frac{df}{d\phi}\right)_0\right].$$

The index 0 denotes that the values of the functions are taken at $\xi = 0$.

Using the boundary conditions (21), we proceed to solve the system (17)–(19) numerically. The behavior of the solution will depend both on the parameters of the polytropic fluid n, σ and the form of the potential energy of the scalar field $\tilde{V}$, and the coupling function f. One of the simplest and most commonly used choices for the potential energy is that of a scalar field with mass m and a quartic self-interaction term ($\kappa > 0$)

$$V = \frac{1}{2}m^2\varphi^2 + \frac{1}{4}\kappa\varphi^4.$$

It was pointed out in [14] that the system with such potential has only singular static solutions. Below, we show that an inclusion of the nonminimal interaction between the scalar field and the polytropic fluid allows one to obtain regular static solutions. As an example, let us choose the coupling function f in dimensionless form as follows:

$$f = \frac{\beta}{2}\phi^2, \qquad \beta > 0. \tag{22}$$

Then, using the dimensionless variables introduced in the previous section, the potential can be rewritten as

$$\tilde{V} = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\Lambda\phi^4, \tag{23}$$

where the new dimensionless constants are given as

$$\mu = \frac{m^2}{\rho_c}\frac{\sigma(n+1)}{4\pi G}, \qquad \Lambda = \frac{\kappa}{\rho_c}\left[\frac{\sigma(n+1)}{4\pi G}\right]^2.$$

Using the above potential $\tilde{V}$ and the function f, Eqs. (17)–(19) take the form

$$\xi^2\frac{1-2\sigma(n+1)v/\xi}{1+\sigma\theta}\frac{d\theta}{d\xi} = \xi^3\left[\frac{\beta}{2}\phi^2\theta^n(1-\sigma\theta) + \mu^2\phi^2 + \frac{1}{2}\Lambda\phi^4 - \frac{1}{\xi^2}\frac{dv}{d\xi}\right] - v, \tag{24}$$

$$\frac{dv}{d\xi} = \frac{\xi^2}{2}\left\{\beta\phi^2\theta^n + \left[1 - 2\sigma(n+1)\frac{v}{\xi}\right] \times \left(\frac{d\phi}{d\xi}\right)^2 + \mu^2\phi^2 + \frac{1}{2}\Lambda\phi^4\right\}, \tag{25}$$

$$\frac{d^2\phi}{d\xi^2} + \left\{\frac{2}{\xi} - \frac{\sigma(n+1)}{1+\sigma\theta}\left[\frac{d\theta}{d\xi} + \frac{1+\sigma\theta}{1-\frac{2\sigma(n+1)v}{\xi}}\frac{1}{\xi}\left(\frac{dv}{d\xi} - \frac{v}{\xi}\right)\right]\right\}\frac{d\phi}{d\xi} = \left[1 - 2\sigma(n+1)\frac{v}{\xi}\right]^{-1}[(\mu^2 - \beta\sigma\theta^{n+1})\phi + \Lambda\phi^3]. \tag{26}$$

The parameter μ can be absorbed by introducing the rescalings $x = \mu\xi$, $\bar{v} = \mu v$, $\bar{\beta} = \beta/\mu^2$, $\bar{\Lambda} = \Lambda/\mu^2$. Bearing this in mind, we assume $\mu = 1$ in further calculations. One can see from these equations that the presence of the interaction between the scalar field and the fluid is defined by the parameter β. In the absence of the fluid, this system only has singular solutions [14]. The inclusion of the fluid changes the situation since the presence of the term $\beta\sigma\theta^{n+1}$ on the right-hand side of (26) for the scalar field corresponds to an effective mass term ($\mu^2 - \beta\sigma\theta^{n+1}$) whose sign depends both on the behavior of the fluid density θ and the values of the parameters β and σ. We note that the present system of gravitating real scalar field plus perfect fluid turns out to be similar to the gravitating complex scalar field system considered in [33]. The gravitating complex scalar field studied in [33] had a potential of the type (23) and it gave regular, *stationary* solutions. Thus, this is already a hint that we can expect regular, stationary solutions for the present system.

IV. NUMERICAL RESULTS

A. Relativistic case

The results of the numerical calculations for the system (24)–(26) with the boundary conditions (21) are presented in Tables I and II. The solutions were started near the origin (i.e. near $\xi \approx 0$) and solved out to a point $\xi = \xi_1$ where the function θ becomes zero. From Eqs. (9) and (10) this is where the fluid vanishes and it is this point, $\xi = \xi_1$, that we define to be the surface of the star. Beyond the point $\xi = \xi_1$, we continued the numerical solutions with only the gravitational and scalar fields while the fluid was set to zero. The interior and exterior solutions were then connected to one another. Details of this are given below. Previous numerical studies with a fluid having a polytropic equation of state [28] found regular, relativistic starlike configurations. These starlike solutions of [28] were found for the values of the parameters $0 \leq \sigma \leq 0.75$ and $1 \leq n \leq 3$. Since our system has two additional parameters (Λ, β), we restrict ourselves to examine just two sets of the parameters (i.e. $\sigma = 0.2$, $n = 1.0$ and $\sigma = 0.2$, $n = 1.5$) at different values of Λ and β when looking for regular solutions. Below, we show that these two sets of solutions differ considerably in the behavior of their physical characteristics.

TABLE I. The parameters of the relativistic configurations with the constant central value of the scalar field $\phi_0 = 0.45$.

Λ	β	ξ_1	$\bar{\xi}_1$	$v(\xi_1)$	$\rho_c/\bar{\rho}$	$-\Omega/M$
			$\sigma = 0.2$, $n = 1.5$			
1500	10 200	0.37	0.4017	0.0269	0.6273	0.1344
1250	8500	0.41	0.4450	0.0295	0.7786	0.1344
1000	6900	0.47	0.5092	0.0326	1.0609	0.1341
750	5200	0.54	0.5851	0.0376	1.3972	0.1341
500	3500	0.66	0.7150	0.0458	2.0939	0.1340
250	1800	0.95	1.0277	0.0637	4.4834	0.1337
150	1130	1.25	1.3494	0.0801	8.1296	0.1331
100	790	1.58	1.7012	0.0955	13.7681	0.1326
75	620	1.83	1.9678	0.1075	19.0029	0.1322
60	525	2.13	2.2830	0.1160	27.7771	0.1316
41	395	2.70	2.8820	0.1330	49.3308	0.1308
30	325	3.35	3.5573	0.1449	86.4977	0.1299
20	260.5	4.70	4.9472	0.1594	217.0848	0.1287
10	195.24	9.00	9.3259	0.1798	1351.2011	0.1269
5	162.1683	21.00	21.4257	0.1983	15 568.1073	0.1229
3	148.834 521 6	30.00	30.4723	0.2102	42 819.2173	0.1192
			$\sigma = 0.2$, $n = 1.0$			
1500.0	8700	0.210	0.2315	0.0313	0.0985	0.1229
1250.0	7250	0.230	0.2535	0.0343	0.1181	0.1229
1000.0	5800	0.257	0.2833	0.0384	0.1473	0.1230
500.0	2910	0.362	0.3992	0.0543	0.2909	0.1231
350.0	2050	0.430	0.4742	0.0648	0.4090	0.1231
200.0	1200	0.567	0.6250	0.0846	0.7181	0.1229
150.0	920	0.655	0.7216	0.0965	0.9712	0.1227
100.0	640	0.800	0.8805	0.1152	1.4810	0.1222
50.0	350	1.110	1.2204	0.1559	2.9250	0.1217
30.0	238	1.400	1.5362	0.1880	4.8644	0.1209
20.0	180	1.650	1.8084	0.2160	6.9318	0.1203
10.0	126	2.160	2.3567	0.2551	13.1668	0.1185
6.8	109	2.442	2.6577	0.2724	17.8189	0.1175
5.0	100	2.670	2.8992	0.2825	22.4626	0.1167
3.0	90	3.000	3.2474	0.2950	30.5069	0.1157
1.0	80	3.500	3.7719	0.3094	46.1887	0.1145
0.3	76	3.670	3.9511	0.3169	52.0002	0.1141
0.1	75	3.750	4.0345	0.3186	55.1796	0.1140

TABLE II. The parameters of the relativistic configurations as a function of the central value of the scalar field ϕ_0 for fixed observable radius of the star $\bar{\xi}_1$ and for given values of σ and n.

ϕ_0	Λ	β	ξ_1	$v(\xi_1)$	$\rho_c/\bar{\rho}$	$-\Omega/M$
$\sigma = 0.2$, $n = 1.5$, $\bar{\xi}_1 = 3.219$						
Relativistic star without a scalar field						
			2.699	0.9604	6.8270	0.2207
Relativistic star with the scalar field						
0.30	2101.0	1642.5	3.055	0.1022	92.9763	0.1325
0.35	617.0	868.0	3.040	0.1186	78.9467	0.1323
0.40	184.0	531.0	3.030	0.1306	70.9796	0.1316
0.44	54.5	385.0	3.025	0.1376	67.0397	0.1308
0.45	34.9	355.0	3.025	0.1396	66.0795	0.1304
0.47	6.6	317.0	3.025	0.1402	65.8043	0.1297
$\sigma = 0.2$, $n = 1.0$, $\bar{\xi}_1 = 2.657$						
Relativistic star without a scalar field						
			2.277	1.1430	3.4430	0.2108
Relativistic star with the scalar field						
0.27	510.15	275.10	2.424	0.2986	15.8983	0.1262
0.30	300.40	225.90	2.425	0.2947	16.1309	0.1250
0.33	178.90	190.30	2.427	0.2903	16.4159	0.1237
0.35	124.50	169.95	2.430	0.2889	16.5532	0.1228
0.39	57.70	142.00	2.436	0.2794	17.2429	0.1207
0.41	36.00	130.00	2.442	0.2769	17.5278	0.1197
0.43	19.30	119.00	2.442	0.2739	17.7255	0.1185
0.45	6.80	109.00	2.442	0.2724	17.8189	0.1175
0.46	1.90	105.00	2.442	0.2714	17.8864	0.1171

The parameters of the system under consideration for the chosen values of σ and n are given in Tables I and II. The procedure for finding these parameters is the following: given the values of σ and n, we seek *eigenvalues* (i.e. values of the parameters Λ and β) for which the function θ goes to zero at some finite value of $\xi = \xi_1$ which as mentioned above we take to correspond to the surface of the star. Then since $M(r) = Mu(r)$ (where M is the total mass of the star between $r = 0$ and its surface at $r = r_b$) it is required that $u(r_b) = 1$. Using this, we can evaluate the function $v(\xi)$ from (16) at $\xi = \xi_1$

$$v(\xi_1) = \frac{A^3 M}{4\pi\rho_c}. \tag{27}$$

This quantity defines the total mass M of the configuration through the parameters of the fluid σ, n and ρ_c which determine the parameter A.

The value ξ_1 of the coordinate ξ, corresponding to the boundary of the configuration, does not represent the radius of the star as measured by a distant observer. To define this radius, it is necessary to make a coordinate transformation to a new dimensionless variable $\bar{\xi}$ which is defined as follows:

$$\bar{\xi} = \int_0^{\xi} e^{\lambda/2} d\xi,$$

or taking into account (14) and (16)

$$\bar{\xi} = \int_0^{\xi} [1 - 2\sigma(n+1)v(\xi)/\xi]^{-1/2} d\xi. \tag{28}$$

Then the observable radius of the configuration $\bar{R}$ in dimensional variables is defined as $\bar{R} = \bar{\xi}_1/A$ in accordance with the data presented in Tables I and II.

Also, the ratio of the central density ρ_c (of the fluid only) to the average density $\bar{\rho}$ (of both the fluid and the scalar field) is presented in the tables. We define the average density as [28]

$$\bar{\rho} = \frac{M}{(4/3)\pi R^3} = \frac{3MA^3}{4\pi\xi_1^3}.$$

Using the expression for $v(\xi)$ from (27), one can express the quantity $MA^3/4\pi$ in terms of the central density ρ_c and the boundary value $v(\xi_1)$ of the mass function. Then we obtain the relation

$$\frac{\rho_c}{\bar{\rho}} = \frac{\xi_1^3}{3v(\xi_1)}.$$

Finally, in the last column of both tables the gravitational potential energy expressed in units of M is shown (see [28] for more details):

$$\frac{\Omega}{M} = 1 - \frac{1}{v(\xi_1)} \int_0^{\xi_1} \frac{T_0^0 \xi^2 d\xi}{[1 - 2\sigma(n+1)v(\xi)/\xi]^{1/2}},$$

where the expression for the energy density is defined from (2). Using Eqs. (14), (16), (22), and (23) one can write the energy density in the following form (in units of ρ_c):

$$T_0^0 = \frac{1}{2}\left\{\beta\phi^2\theta^n + \left[1 - 2\sigma(n+1)\frac{v}{\xi}\right]\left(\frac{d\phi}{d\xi}\right)^2 + \mu^2\phi^2 + \frac{1}{2}\Lambda\phi^4\right\}. \tag{29}$$

The absolute value of the potential energy Ω represents the work that would have to be done on the system to diffuse its mass to infinity.

Next, using the data from the tables, in Fig. 1 we plot $v(\xi_1)$, which is proportional to the total mass of the configuration M, on the value of the self-coupling parameter Λ at some constant central $\phi_0 = 0.45$. In Fig. 2 we plot $v(\xi_1)$ as a function of the value of central ϕ_0 for different n values and $\sigma = 0.2$. From Fig. 1 one can see that, while Λ increases, the masses of the configurations decrease both for $n = 1.0$ and $n = 1.5$. Initially, for small Λ, the masses are very different for $n = 1.0$ and $n = 1.5$; asymptotically, for large Λ, they approach comparable values. This is because for large Λ, the total mass is defined by the scalar field but not by the contribution from the fluid. The sizes of the configurations are different for the two cases $n = 1.0$ (which has a size characterized by $\bar{\xi} = 2.657$) and $n = 1.5$ (which has a size characterized by $\bar{\xi} = 3.219$).

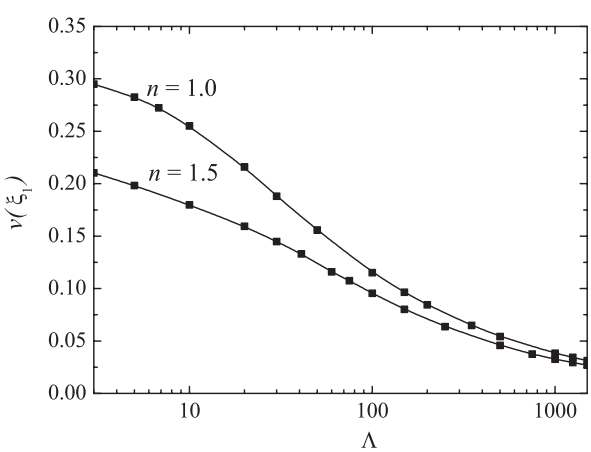

FIG. 1. The dependence of the function $v(\xi_1)$ (which is proportional to the total mass M) on the value of the coupling parameter Λ from (23) at the fixed central value $\phi_0 = 0.45$ and $\sigma = 0.2$. The data are taken from Table I.

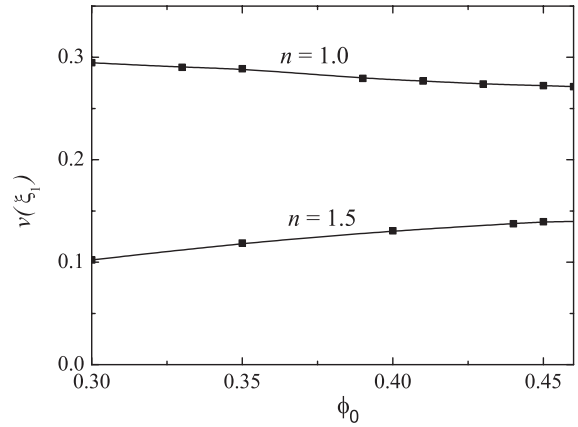

FIG. 2. The dependence of the function $v(\xi_1)$ on the central value of ϕ_0 at the fixed $\bar{\xi}_1 = 3.219$ for $n = 1.5$ and $\bar{\xi}_1 = 2.657$ for $n = 1.0$, and $\sigma = 0.2$ for both graphs. The data are taken from Table II.

On the other hand, if one fixes the size of the star to equal the size of a corresponding relativistic configuration without a scalar field, and changes the central value of ϕ_0 so as to get regular solutions, it is necessary to choose appropriate eigenvalues of Λ and β. In this case, $v(\xi_1)$ as a function of ϕ_0 is given in Fig. 2. From this figure one sees that as ϕ_0 increases, the mass of the configuration can either increase (for the $n = 1.5$ case) or decrease (for the $n = 1.0$ case).

It follows from Tables I and II and Figs. 1 and 2 that the masses of the configurations, computed for the values of the system parameters used here, are considerably smaller than the masses of the stars without a scalar field. The maximal dimensionless masses of the configuration *with* the scalar field for $n = 1.0$ and at $n = 1.5$ are 0.32 and 0.21, respectively; *without* a scalar field the masses of the same configurations are 1.14 and 0.96, respectively. Our attempts to increase the masses of the system by varying ϕ_0 were not successful. From Fig. 2, the implication is that the masses change only slightly with ϕ_0.

Next, in Fig. 3 the metric functions e^λ, e^ν, and the mass distribution $M(\bar{\xi}/\bar{\xi}_1)$ (as a function of the dimensionless radius $\bar{\xi}$) are given. In plotting these functions, we used the following procedure:

Inside the Star: The function e^ν was plotted using (13); the function e^λ was plotted from Eq. (14), in terms of the dimensionless variables from (16). Explicitly,

$$e^\lambda = \left[1 - 2\sigma(n+1)\frac{v}{\xi}\right]^{-1}. \tag{30}$$

From the interior solution, parts of Fig. 3 (i.e. the region $0 < \xi < \xi_1$) we see the reason for the terminology "chameleon star" for the present solutions – the scalar field mimics the behavior of the fluid in the interior region. Even in the exterior region where the fluid goes to zero, the scalar field is asymptotically going to zero.

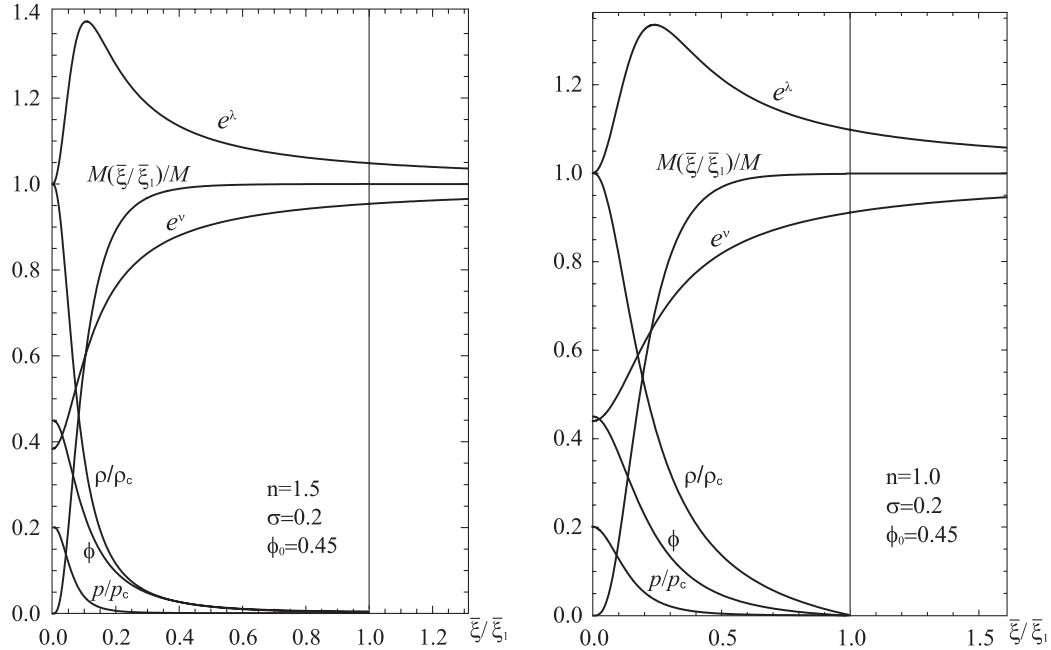

FIG. 3. The density ρ and the pressure p of the fluid, the scalar field ϕ, the metric functions e^ν, e^λ and the current mass M as functions of the relative invariant radius $\bar\xi/\bar\xi_1$ for $n = 1.5$ and $n = 1.0$ respectively. To provide the asymptotic flatness of the solutions, i.e. e^ν, $e^\lambda \to 1$ as $\bar\xi \to \infty$, the value of the constant ν_c needed to be chosen as $\nu_c \approx -0.958$ for $n = 1.5$ and $\nu_c \approx -0.822$ for $n = 1.0$.

Outside the star: The solution goes to the Schwarzschild solution

$$e^\nu = e^{-\lambda} = 1 - 2\sigma(n+1)\frac{v(\xi_1)}{\xi}. \tag{31}$$

The mass function is defined as

$$M(\bar\xi/\bar\xi_1) = M\frac{v(\xi)}{v(\xi_1)}.$$

Finally, in Figs. 4 and 5 we plot the energy density, T_0^0, from the expression (29). The definition of the mass function given just above takes into account the energy of the fluid and the scalar field from $\xi = 0$ to $\xi = \xi_1$. Although the fluid vanishes at $\xi = \xi_1$ the scalar field does not vanish. Thus, in principle one should include the contribution to the mass function of the scalar field from $\xi = \xi_1$ to $\xi \to \infty$. However, from the Figs. 3–5 one can see that ϕ and $\phi\prime$ rapidly go to zero as $\xi \to \infty$ and thus the mass function is given essentially just by the fluid and scalar field energy density between $\xi = 0$ and $\xi = \xi_1$.

We now turn the question of connecting the interior region (i.e. $0 < \xi < \xi_1$ where one has both fluid plus scalar field as a source) with the exterior region (i.e. $\xi \geq \xi_1$ where one has only a rapidly vanishing scalar field). For this purpose, we write down the Einstein equations of (4) and (5) and the scalar field equation of (6), without the fluid source, i.e. $\theta = 0$ so that from (9) and (10) we have zero pressure and density. This leads to the following system of equations:

$$-e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} = 8\pi G\left[\frac{1}{2}e^{-\lambda}\varphi'^2 + V(\varphi)\right], \tag{32}$$

$$-e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) + \frac{1}{r^2} = 8\pi G\left[-\frac{1}{2}e^{-\lambda}\varphi'^2 + V(\varphi)\right], \tag{33}$$

$$\varphi'' + \left[\frac{2}{r} + \frac{1}{2}(\nu' - \lambda')\right]\varphi' = e^\lambda\frac{dV}{d\varphi}, \tag{34}$$

which, by analogy with the transformations made above, can be rewritten in terms of the dimensionless variables $v(\xi)$, $\phi(\xi)$ and $\nu(\xi)$ as follows:

$$\frac{dv}{d\xi} = \xi^2\left[\frac{1}{2}e^{-\lambda}\left(\frac{d\phi}{d\xi}\right)^2 + \tilde V\right], \tag{35}$$

$$\frac{d\nu}{d\xi} = \frac{2\sigma(n+1)e^\lambda}{\xi}\left[\frac{v}{\xi} + \xi^2\left(\frac{1}{2}e^{-\lambda}\left[\frac{d\phi}{d\xi}\right]^2 - \tilde V\right)\right], \tag{36}$$

$$\frac{d^2\phi}{d\xi^2} + \frac{2}{\xi}\left\{1 + \frac{\sigma(n+1)e^\lambda}{2}\left[\frac{2v}{\xi} + \xi^2\left(\frac{1}{2}e^{-\lambda}\left[\frac{d\phi}{d\xi}\right]^2 - \tilde V\right) - \frac{dv}{d\xi}\right]\right\}\frac{d\phi}{d\xi} = e^\lambda\frac{d\tilde V}{d\phi}, \tag{37}$$

where e^λ is given by (30), and $\tilde V$ is taken from (23). Note that this system of equations (32)–(34) (or (35)–(37)) is essentially the same as those obtained in [14] which considered a real scalar field with quartic self-interaction. The above system contains the parameter σ as a trace of the influence of the fluid on the external solution. The solution sought beginning from the surface of the star at $\xi = \xi_1$

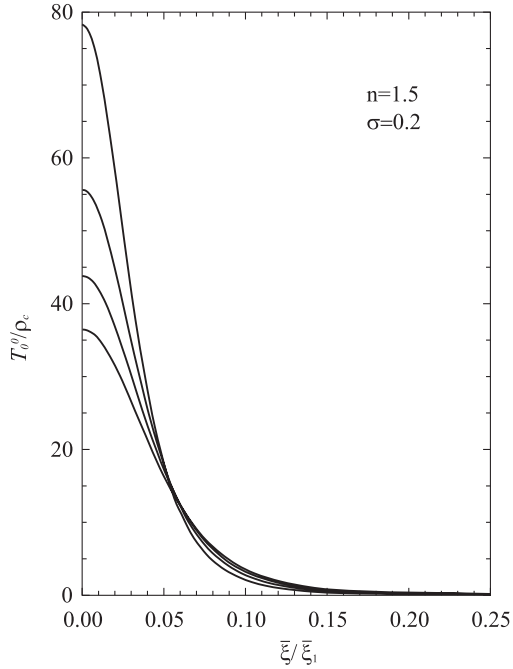

FIG. 4. The distributions of the total energy density in units of ρ_c from (29) for different the initial values of $\phi_0 = 0.30$, 0.35, 0.40, 0.45, from top to bottom.

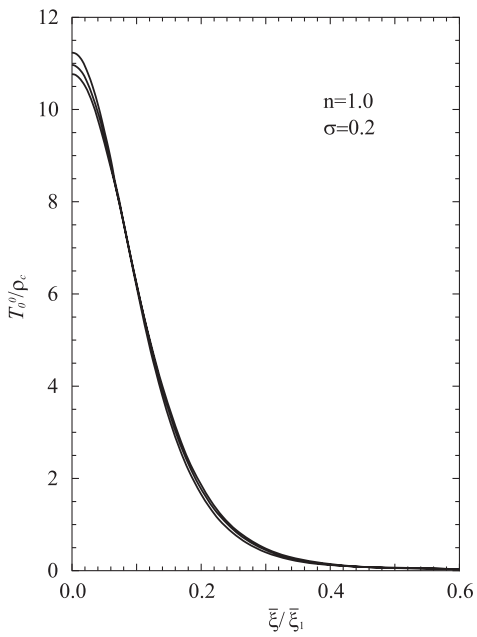

FIG. 5. The distributions of the total energy density in units of ρ_c from (29) for the different initial values of $\phi_0 = 0.45$, 0.35, 0.27, from top to bottom.

using, as the boundary conditions, the values of $\upsilon(\xi_1)$, $\varphi(\xi_1)$ and $\nu(\xi_1)$ obtained from the solution of Eqs. (24)–(26) for the internal part of the configuration. This allows one to determine the value of the integration constant ν_c from (13) by requiring e^ν to be equal to unity at infinity, providing asymptotical flatness of the space-time. (The values of ν_c for the examples shown in Fig. 3 are given in the caption.) Thus, the complete solution for the configuration under consideration is derived by matching of the internal fluid solutions given by Eqs. (24)–(26) with the external solutions obtained from the system (35)–(37).

The system (35)–(37) has obvious asymptotically flat solutions in the form $\upsilon \to \text{const} \approx \upsilon(\xi_1)$, with $\upsilon(\xi_1)$ taken from (31) that corresponds to the fact that, by choosing the eigenvalues of the parameters Λ and β presented in Tables I and II, the external scalar field makes negligible contribution to the total mass of the configuration. It allows using the solution for the metric functions in the form of (31) as a good approximation; $\nu \to \text{const}$, where the constant is made equal to zero by the corresponding choice of ν_c (see above); $\phi \to \text{const} e^{-\mu\xi}/\xi$. The numerical calculations of the system (35)–(37) confirm this asymptotic behavior.

From Figs. 3–5, we can draw the following conclusions about the metric functions, e^λ, e^ν and the mass distribution $M(\bar{\xi}/\bar{\xi}_1)$:

(1) Asymptotically, as $\xi \to \infty$, the space-time becomes flat, i.e. e^ν, $e^\lambda \to 1$. To get this asymptotic behavior of the metric function e^ν, it is necessary to choose a specific value for the central value, ν_c (see the caption of Figs. 3). The asymptotic behavior of $e^\lambda \to 1$ follows from Eq. (30) and the fact that $\upsilon(\xi)|_{\xi\to 0} \to 0$ as ξ^3.

(2) As shown in [28], the relativistic configurations without a scalar field are characterized by a greater concentration of matter toward the center than in the nonrelativistic case. By including a scalar field, we obtain even greater concentration of matter toward the center. This can be seen by comparing Fig. 3 with Figs. 1 and 2 from the paper [28]. The main reason for the increased concentration of mass is the nonminimal coupling between the scalar field and the fluid with the choice of the coupling function given by (22). The functional form of this coupling gives the required regular solutions, lying in the range $0 < \xi < \xi_1$, only for the eigenvalues of the parameter $\beta \gg 1$ (see Tables I and II). This leads to the fact that near the origin, when $\xi \approx 0$, a greater concentration of mass occurs due to the presence of the term $\beta\phi^2\theta^n$ in the expression for the energy density (29). At the same time, both ϕ and ϕ' give relatively small contributions to the energy density compared to the term coming from the nonminimal scalar-fluid coupling. This fact will become important, below, when we consider the stability of the solutions.

(3) The total energy density presented in Figs. 4 and 5 strongly depends on the polytropic index n. At comparable central values of the scalar field ϕ_0, the energy density for the case $n = 1.5$ is several times greater than for the case $n = 1.0$. In turn, both

values of n yield a greater energy density at the center of the configurations than in the case of the relativistic stars without a scalar field [28] when the energy density was found to be 1.

(4) Despite the higher concentration of matter at the center of the configurations with the addition of the scalar field, their masses are considerably smaller than the masses of the relativistic stars of the same size but without a scalar field (see Table II). This occurs as a consequences of the fact that the external region of the stars, with only the scalar field as a source, is strongly rarefied due to the rapid vanishing of the scalar field in the region $\xi \geq \xi_1$.

B. Nonrelativistic case

In this section, we consider the nonrelativistic limit of the system (24)–(26). The nonrelativistic limit corresponds to $\sigma \to 0$, and $p \ll \rho$. First, we recall some results about this system in the absence of the scalar field [32]. With no scalar field, the system of Eqs. (24)–(26) reduces to the well-known Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right] = -\theta^n. \tag{38}$$

This equation has solutions which describe finite size configurations for different values of the parameter n [32]. The nonrelativistic limit for the case when there a scalar field is obtained by omitting all terms with σ in Eqs. (24)–(26) except for the term $\beta\sigma\theta^{n+1}$ in the scalar field Eq. (26). This term must be kept since the product $\beta\sigma$ is generally a nonzero quantity even for $\sigma \to 0$. Taking all the above into account, we rewrite the system (24)–(26) in the following form:

$$\xi^2\frac{d\theta}{d\xi} = \xi^3\left[\frac{\beta}{2}\phi^2\theta^n + \mu^2\phi^2 + \frac{1}{2}\Lambda\phi^4 - \frac{1}{\xi^2}\frac{dv}{d\xi}\right] - v, \tag{39}$$

$$\frac{dv}{d\xi} = \frac{\xi^2}{2}\left\{\beta\phi^2\theta^n + \left(\frac{d\phi}{d\xi}\right)^2 + \mu^2\phi^2 + \frac{1}{2}\Lambda\phi^4\right\}, \tag{40}$$

$$\frac{d^2\phi}{d\xi^2} + \frac{2}{\xi}\frac{d\phi}{d\xi} = (\mu^2 - \beta\sigma\theta^{n+1})\phi + \Lambda\phi^3. \tag{41}$$

Here, as in the relativistic case, the presence of the term $\beta\sigma\theta^{n+1}$ is important—it leads to a change in sign of the effective mass term $(\mu^2 - \beta\sigma\theta^{n+1})$ for certain values of β. This feature is important for the existence of regular solutions.

Proceeding as in the previous section, we obtain numerical results for the nonrelativistic limit which we present in Table III. We chose the parameter σ to be $\sigma = 0.001$. Such a small value of σ requires that β be large enough so that the first term on the right-hand side of Eq. (41) is negative, which is the necessary condition to have regular solutions. In the nonrelativistic case, as in the relativistic case, we need to find eigenvalues of two parameters, β and Λ, in order to have regular solutions for a given values of the polytropic index, n, and the central value of the scalar field, ϕ_0. From Table III we do indeed find that there are values of β large enough to make the first term on the right-hand side of (41) negative, thus yielding regular solutions.

Using the data from Table III, the dependence of the $v(\xi_1)$ on the central value of the scalar field ϕ_0 is given in Fig. 6. Figure 7 uses the date from Table III to plot the ratio of the energy density to the central density, T_0^0/ρ_c as a function of the dimensionless, normalized radius. Also from Table III one can see that we have fixed the radius of a star $R = \xi_1$ to be equal to the size of a nonrelativistic star without a scalar field. From the figure we can draw the following observations:

(1) The masses of the stars with a scalar field are considerably smaller than the masses of the stars without a scalar field. In the case $n = 1.0$ the masses are 4–5 times smaller, and in the case $n = 1.5$ about 6 times smaller.

(2) The mass is a fairly slowly varying function of ϕ_0, especially when $n = 1.5$.

(3) There is a greater concentration of the mass density towards the center of the stellar configurations. This can be seen explicitly by comparing with the distribution of the mass density for a star without a scalar field which is also presented in Fig. 7.

All these results are very similar to those of the relativistic case.

C. Stability of the solutions

In this section, we discuss the issue of the stability of the regular solutions obtained above. There are two basic approaches to studying the stability: (i) The energy approach to the theory of equilibrium for a star [32]; (ii) A more rigorous dynamical stability approach based on studying the stability of linear and nonlinear time-dependent perturbations. In this paper we will use the first approach. Proceeding along the lines of reference [28], we define the total energy E of the system, including the internal and gravitational energies, as

$$E = M = 4\pi\int_0^R T_0^0 r^2 dr, \tag{42}$$

where T_0^0 is defined by (2) and corresponds to the total energy density of the system (recall that $c = 1$). Next, we consider a system consisting of a gas of particles having a rest-mass density ρ_g. Its relativistic energy density is the sum of the rest energy ρ_g plus the density of internal energy. For the special case of an adiabatic process which assumes the absence of heat flow terms in the energy-momentum tensor (2), it is possible to obtain a relation between the gas density ρ_g and the total mass density $\rho_t \equiv T_0^0$ or alternatively between ρ_g and θ, φ.

TABLE III. The parameters of the nonrelativistic configurations with a variable central value of the scalar field ϕ_0. The observable radius of a star $\bar{\xi}_1$ is fixed. The choice of the parameters σ and n is indicated in the table.

ϕ_0	Λ	β	ξ_1	$v(\xi_1)$	$\rho_c/\bar{\rho}$	$-\Omega/M$
$\sigma = 0.001$, $n = 1.5$, $\bar{\xi}_1 = 3.6560$						
Nonrelativistic star without a scalar field						
			3.6538	2.7141	5.9907	0.0000
Nonrelativistic star with the scalar field						
0.039	1040	20 600	3.6538	0.4384	37.0928	−0.0009
0.038	1850	21 800	3.6538	0.4373	37.1840	−0.0009
0.036	3980	24 500	3.6538	0.4354	37.3411	−0.0009
0.034	7010	27 600	3.6538	0.4346	37.4105	−0.0009
0.030	17 575	35 500	3.6538	0.4349	37.3895	−0.0009
0.028	25 895	39 500	3.6538	0.4462	36.4388	−0.0009
0.027	31 270	41 700	3.6538	0.4556	35.6855	−0.0009
$\sigma = 0.001$, $n = 1.0$, $\bar{\xi}_1 = 3.1437$						
Nonrelativistic star without a scalar field						
			3.1416	3.1416	3.2899	0.0000
Nonrelativistic star with the scalar field						
0.039	190	10700	3.1416	0.6568	15.7364	−0.0008
0.038	595	11000	3.1416	0.6687	15.4556	−0.0008
0.036	1645	12000	3.1416	0.6761	15.2872	−0.0008
0.034	3070	13100	3.1416	0.6868	15.0493	−0.0008
0.030	7645	15700	3.1416	0.7166	14.4227	−0.0008
0.025	20685	21000	3.1416	0.7457	13.8609	−0.0008

To do this, we use the first law of thermodynamics which, in our case, takes the form

$$d\rho_t + (\rho_t + p_t)\frac{dV}{V} = 0, \tag{43}$$

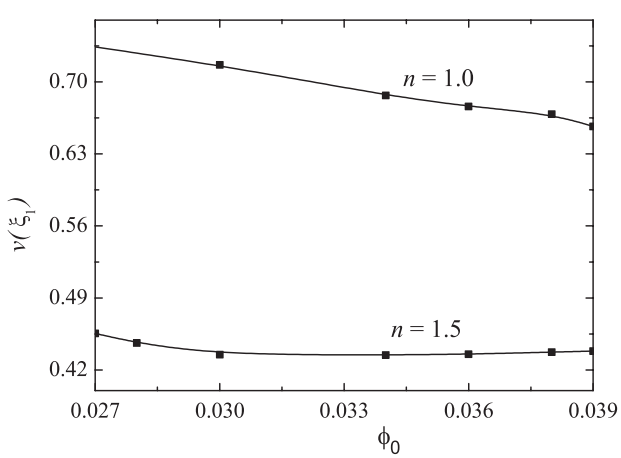

FIG. 6. The dependence of the function $v(\xi_1)$ on the central value of ϕ_0 at the fixed $\bar{\xi}_1 = 3.6560$ for $n = 1.5$ and $\bar{\xi}_1 = 3.1437$ for $n = 1.0$, and $\sigma = 0.001$ for both graphs. The data are taken from Table III.

where $p_t \equiv -T_1^1$ is the total pressure, and V is the specific volume (this symbol V should not be confused with the potential energy used earlier). Since $dV/V = -d\rho_g/\rho_g$, Eq. (43) gives the following relation:

$$\frac{d\rho_g}{\rho_g} = \frac{d\rho_t}{\rho_t + p_t}. \tag{44}$$

In general, when ρ_t and p_t are functions of θ, φ, this equation cannot be integrated. However, for our configuration, the numerical calculations performed in the previous sections show that the main part of the energy is provided by the term containing the nonminimal coupling but not by φ' and $V(\varphi)$. This allows us to neglect the terms containing the scalar field kinetic energy and potential as compared with $f\rho$ and fp in the T_0^0 and T_1^1 components of the energy-momentum tensor (2), respectively. In this approximation we have the following expressions for the total mass density and the pressure:

$$\rho_t \approx f\rho, \qquad p_t \approx fp.$$

Substituting these expressions into (44) and taking into account that $\rho = \rho_c\theta^n$ and $p = \sigma\rho_c\theta^{n+1}$, we have from (44)

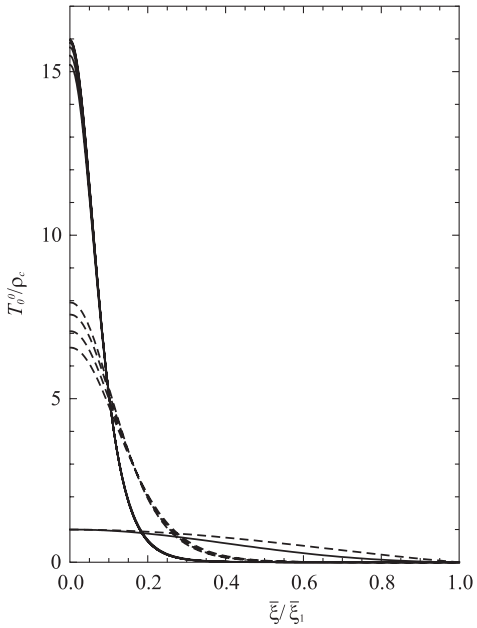

FIG. 7. The plot of the total energy density, T_0^0, in units of ρ_c in the nonrelativistic case ($\sigma = 0.001$) for different central values of ϕ_0 from Table III. The solid lines correspond to the case of $n = 1.5$, the dashed lines to the case of $n = 1.0$. The top two sets of curves correspond to the configurations with the scalar field. For comparison, the energy densities of the nonrelativistic configurations without a scalar field $T_0^0 = \rho_c\theta^n$ are also shown by the two bottom, less peaked curves.

$$\frac{d\rho_g}{\rho_g} = \frac{nd\theta}{\theta(1+\sigma\theta)} + \frac{df}{f(1+\sigma\theta)}.$$

This equation differs from the case without a scalar field by the presence of the second term on the right-hand side containing the coupling function f. In the absence of the nonminimal coupling, i.e. when $f = 1$, this term vanishes. Integrating, we find

$$\rho_g = \rho_{gc}\left[\frac{(1+\sigma)\theta}{1+\sigma\theta}\right]^n \exp\left[\int \frac{df}{f(1+\sigma\theta)}\right], \tag{45}$$

where ρ_{gc} is the value of the gas density ρ_g at the center of the configuration. This constant may be evaluated as follows [28]: near the boundary of the configuration, where $\theta \ll 1$, Eq. (45) becomes approximately

$$\rho_g \approx \rho_{gc}(1+\sigma)^n\theta^n \exp\left(\int_{\xi_a}^{\xi} \frac{df}{f}\right),$$

where ξ_a is the point starting from which the approximation $\theta \ll 1$ becomes valid. This equation can be rewritten in differential form as

$$\frac{d}{d\xi}\left\{\ln\left[\frac{\rho_g}{\rho_{gc}(1+\sigma)^n\theta^n}\right]\right\} \approx \frac{d}{d\xi}\left(\int_{\xi_a}^{\xi} \frac{df}{f}\right). \tag{46}$$

Since $f = f(\phi(\xi))$ is a function of ξ, the term on the right-hand side can be evaluated as follows:

$$\frac{d}{d\xi}\left(\int_{\xi_a}^{\xi} \frac{df}{f}\right) \equiv \frac{d}{d\xi}\left(\int_{\xi_a}^{\xi} \frac{(df/d\xi)d\xi}{f}\right) = \frac{d}{d\xi}(\ln f).$$

Taking this expression into account, Eq. (46) takes the form

$$\rho_g \approx \rho_{gc}(1+\sigma)^n f\theta^n.$$

Near the boundary, the internal energy density is small compared to the rest-mass-energy density, so that $\rho_t \approx \rho_g$. Comparing the above equation with the expression $\rho_t \approx f\rho_c\theta^n$, we obtain

$$\rho_{gc} = \frac{\rho_c}{(1+\sigma)^n},$$

and Eq. (45) becomes

$$\rho_g = \rho_c\left[\frac{\theta}{1+\sigma\theta}\right]^n \exp\left[\int \frac{df}{f(1+\sigma\theta)}\right]. \tag{47}$$

Using expression (47), the corresponding proper energy E_{0g} of the gas may be defined as the integral of ρ_g with respect to proper volume $dV = 4\pi r^2 e^{\lambda/2}dr$ for the metric (3) as follows:

$$E_{0g} = M_{0g} = 4\pi\int_0^R \rho_g e^{\lambda/2} r^2 dr. \tag{48}$$

The quantity M_{0g} is proportional to the total number of particles N in the configuration, $M_{0g} \equiv N/A$, where A is Avogadro's number. Using the dimensionless variables (16) and expressions (30) and (47), we obtain the proper energy of the gas (48) in units of the total energy $E = M$ in the following form:

$$\frac{E_{0g}}{E} = \frac{1}{v(\xi_1)}\int_0^{\xi_1} \frac{\theta^n \exp\{\int_0^{\xi_1}[f(1+\sigma\theta)]^{-1}df\}\xi^2 d\xi}{[1+\sigma\theta]^n[1-2\sigma(n+1)v/\xi]^{1/2}}. \tag{49}$$

This expression differs from the corresponding expression in [28] through the presence of an extra factor exp{...} under the integral which comes from the nonminimal coupling in the system. Stability of the configuration can be tested via the sign of the expression [28,32]

$$\frac{\text{Binding Energy}}{E} = \frac{E_{0g} - E}{E} = \frac{E_{0g}}{E} - 1. \tag{50}$$

The necessary (but not a sufficient) condition for stability of the system that $E_{0g} > E$ i.e. that the total system energy E is less than the energy of the nongravitationally interacting gas, E_{0g}, making E the energetically preferred state. The condition in (50) amounts to requiring that the Binding Energy defined in this equation be positive. In the paper [34], the question of stability of adiabatic polytropic configurations was considered. It was shown that such configurations may have both positive and negative binding energies depending on the value of the polytropic index n,

and the parameter σ (configurations with $1 \leq n \leq 3$ and $0.01 \leq \sigma \leq 100$ were considered).

When a nonminimally coupled scalar field is included, we considered a very narrow range of these parameters, $\sigma = 0.2$ and $n = 1.0, 1.5$, and also only one choice of the coupling function f given by the form (22). In this case, the numerical calculations indicate that $E_{0g}/E \ll 1$, i.e. we have the negative binding energy, and correspondingly the configurations under consideration are unstable. The same values of σ and n without a scalar field give configurations with the positive binding energy [34]. Obviously, the difference between the present results and those of [34] is connected with the presence of the extra factor exp{...} under the integral in the expression (49) whose value, as numerical calculations indicate, is much less than unity for the parameters σ, n used in the paper, and for the coupling function f of the form (22).

The behavior of exp{...} in (49) can be approximately estimated as follows: Since we are looking only for solutions with $0 \leq \theta \leq 1$, the value of $(1 + \sigma\theta)$ may be approximated as giving a finite contribution to the value of integral $\int_0^{\xi_1}[f(1+\sigma\theta)]^{-1}df$ in the form of some constant factor $\alpha > 0$, i.e. $\int_0^{\xi_1}[f(1+\sigma\theta)]^{-1}df \approx \alpha \int_0^{\xi_1} d\ln f$. Then one can see that

$$\exp\left\{\int_0^{\xi_1}[f(1+\sigma\theta)]^{-1}df\right\} \approx \left[\frac{f(\xi_1)}{f(0)}\right]^{\alpha}.$$

When the coupling function f is taken in the form (22), we have the finite value of $f(0)$, and since $\phi(\xi_1) \to 0$ then $f(\xi_1) \to 0$ as well, thereby suppressing the value of E_{0g}/E in (49). In the nonrelativistic case, where $\sigma \to 0$, we have $\alpha \to 1$ and the situation with the instability is the same.

Possible ways of resolving this problem are:

(i) Consider configurations for which $\phi(\xi_1) \neq 0$, and the function $f(\xi)$ varies slowly in the range $0 < \xi < \xi_1$ to provide a large value of the above exp{...}. In this case, one can expect that the configuration will have a "tail" of a scalar field outside the fluid at $\xi > \xi_1$ (a similar thing happens for configurations considered in [5]). In this case, it will be necessary to perform a stability analysis of both internal solutions (where the fluid and the scalar field are involved) and external solutions (where only the scalar field is involved).

(ii) Since the solutions and their stability are sensitive to the nature of the scalar field-fluid coupling, which in this paper was take to have the form given in (22), one could look for stable solutions by examining different forms of this coupling; for example, one could try a nonpolynomial coupling of the form $f = e^{\phi}$. These stability studies will be the focus of future work.

V. NONRELATIVISTIC CASE: AN ANALYTICAL SOLUTION FOR A MASSLESS SCALAR FIELD

The numerical results obtained in the previous sections demonstrate the possibility of obtaining the regular solutions both in relativistic and nonrelativistic cases when the coupling function f is chosen in the form of (22). This choice is not the only one possible. It will be shown latter in this section that, by choosing the special form of the function f, it is possible to find an analytical solution in a particular nonrelativistic case when the polytropic index $n = 0$, and the scalar field is taken to be massless. This case corresponds to an incompressible fluid with a constant mass density, $\rho = \rho_c = \text{const}$, and a spatial varying pressure, $p \neq \text{const}$.

To begin, we rewrite the system of Eqs. (24)–(26) for the massless case and in the nonrelativistic limit ($\sigma \to 0$) in the form

$$\xi^2\frac{d\theta}{d\xi} = -\frac{\xi^3}{2}\left(\frac{d\phi}{d\xi}\right)^2 - v, \tag{51}$$

$$\frac{dv}{d\xi} = \xi^2\left[f\theta^n + \frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2\right], \tag{52}$$

$$\frac{d^2\phi}{d\xi^2} + \frac{2}{\xi}\frac{d\phi}{d\xi} = -\sigma\theta^{n+1}\frac{df}{d\phi}. \tag{53}$$

We have kept the σ-containing-term on the right-hand side of Eq. (53) since it is does not necessarily small for *arbitrary* f. Differentiating the first equation of the system, and substituting the second and the third equations into this expression, gives

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -f\theta^n + \sigma\xi\theta^{n+1}\frac{df}{d\xi}. \tag{54}$$

The last term of this equation contains the factor $df/d\xi \equiv \phi' df/d\phi$. For $f = 1$, Eq. (54) reduces to the Lane-Emden Eq. (38) which has an analytical solution for the case of an incompressible fluid with a constant mass density $\rho = \rho_c = \text{const}$, $p \neq \text{const}$, $n = 0$—see [32], Eq. (10.3.14). This solution is

$$\theta = 1 - \frac{1}{6}\xi^2. \tag{55}$$

Equation (54) can also be integrated analytically by choosing the function f to have the following power-law form:

$$f = f_0 + \frac{\beta}{m}\xi^m, \tag{56}$$

where f_0, β and m are arbitrary parameters. Using this function and taking the parameter m to be of order 1 so that the product $\sigma\theta \ll |1/m|$, we have from (54)

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right] = -\left(f_0 + \frac{\beta}{m}\xi^m\right)\theta^n.$$

When $n = 0$, this equation can be integrated giving the following regular solution:

$$\theta = \theta_0 - \frac{f_0}{6}\xi^2 - \frac{\beta}{m(m+2)(m+3)}\xi^{m+2}. \tag{57}$$

Here, θ_0 is an integration constant. A second integration constant was set equal to zero to make the solution regular at $\xi = 0$. We take $\theta_0 = 1$ so that in the absence of the scalar field (i.e. when $f_0 = 1$ and $\beta = 0$) Eq. (57) will reduce to Eq. (55). This solution is regular everywhere, including the point $\xi = 0$, when $m > -2$. The degenerate case $m = 0$ corresponds to the classical solution (55) which can be obtained from (57) by setting $f_0 = 1$ and $\beta = 0$. This choice corresponds to turning Eq. (54) into the Lane-Emden Eq. (38).

We now analyze the behavior of the solution as $\xi \to 0$. To begin, we calculate the first and second derivatives of θ:

$$\frac{d\theta}{d\xi} = -\frac{f_0}{3}\xi - \frac{\beta}{m(m+3)}\xi^{m+1},$$
$$\frac{d^2\theta}{d\xi^2} = -\frac{f_0}{3} - \frac{\beta(m+1)}{m(m+3)}\xi^{m}.$$

For a spherically symmetric solution with maximal mass density as $\xi \to 0$, two conditions must be satisfied: (1) the first derivative must be equal to zero; (2) the second derivative must be negative. The former condition gives $m > -1$. Given this, the behavior of the second derivative is

$$\text{if } (-1 < m < 0)\text{: } \theta'' \to \begin{cases} +\infty & \text{for } \beta > 0; \\ -\infty & \text{for } \beta < 0; \end{cases}$$
$$\text{if } m > 0\text{: } \theta'' \to -f_0/3 \quad \text{for any } \beta.$$

One can see that for $(-1 < m < 0)$ the second condition is satisfied only if $\beta < 0$.

On the other hand, the regularity of the solution assumes the presence of a point on the axis, ξ, where $\theta = 0$. For the classical solution (55), this point is $\xi_1|_{\theta=0} = \sqrt{6}$. In the case of configurations with a scalar field, the location of the point $\xi_1|_{\theta=0}$ will depend on the factors in front of the ξ^{m+2} term in (57), i.e. it will be determined by the values of β and the denominator $m(m+2)(m+3)$. In the range $(-1 < m < 0)$ only negative β are allowed if one wants $\theta'' < 0$ as $\xi \to 0$, which is one of conditions that the solution (57) be regular. In the range $m > 0$, the regularity of the solution is guaranteed if $\beta > 0$, since otherwise the solution grow as a power law of ξ.

The above analysis shows that the inclusion of a scalar field decreases the size of configurations, as compared to the classical case (55), for all acceptable values of the parameter m. In this sense, the behavior of the solutions with a massless scalar field and the coupling function in the form of (56) differs from the behavior of the solutions obtained in Sec. IV B when the function f was chosen to have the form (22). In the latter case, the size of the configurations for given values of n and σ depends considerably on the parameters Λ, β, and ϕ_0, and it can be larger or smaller than the size of a configuration without a scalar field.

Substituting the expression for θ from (57) into the scalar field Eq. (53) with $n = 0$, we find the following analytical solution for the scalar field ϕ in the interior of the configuration

$$\phi_{\rm in} = \phi_0 + \frac{\beta\sigma}{6}\left[\frac{3\beta\xi^{2+m}}{m(m+2)^2(m+3)(2m+3)} + \frac{f_0\xi^2}{(m+3)(m+4)} - \frac{6}{(m+1)(m+2)}\right]\xi^{m+1}. \tag{58}$$

One can see from this expression that the behavior of the scalar field is defined in large measure by the factor $\beta\sigma$ in front of the square brackets. Since in the nonrelativistic limit $\sigma \to 0$, this factor will differ appreciably from zero only for large β. For β not large, one can expect that the field will be practically constant and equal to the central value ϕ_0 up to the boundary of the fluid at $\xi = \xi_1$ where $\theta = 0$. The general solution for ϕ for all ξ is obtained by matching of the internal solution (58) with an external solution of Eq. (53) where the right-hand side is equal to zero. The exterior equation has a solution of the form

$$\phi_{\rm ext} = C_1 + C_2/\xi,$$

where C_1, C_2 are integration constants which are determined by matching of the external solution $\phi_{\rm ext}$ with the interior solution $\phi_{\rm in}$ solution at the boundary of the fluid at $\xi = \xi_1$. One can see from this expression that the corresponding mass density, which is proportional to $\phi_{\rm ext}'^2$, tends asymptotically to zero.

VI. CONCLUSION

In this article we studied gravitating, spherically symmetric, starlike, configurations with a matter source consisting of a normal (i.e. nonghost, nonphantom) scalar field plus a perfect isotropic fluid. The motivation for studying such a model is that scalar fields have found broad use in various cosmological models as well as astrophysical models. It is natural to postulate that, if scalar fields do really play a role in the Universe, these scalar fields might play a role in the structure of compact objects such as main sequence stars or neutron stars. Proceeding from this assumption, we studied a model of a starlike configuration supported by a scalar field nonminimally coupled to ordinary matter in the form of a perfect fluid. As an example, we considered the case when the scalar field had a quadratic mass term and a quartic self-interaction giving a scalar field potential of the form (23). The coupling between the scalar field and the perfect fluid, f, was taken to have the form (22). For this model we studied solutions both in the relativistic and nonrelativistic limit. Our investigation showed that the existence of regular solutions is

possible exactly because of the presence of the nonminimal coupling between the scalar field and fluid. In the absence of such coupling, the potential (23) gives only singular solutions [14]. From the mathematical point of view, the existence of regular solutions in our model is possible because of the appearance in scalar field Eq. (26) of an effective mass term $m_{\rm eff} = (\mu^2 - \beta\sigma\theta^{n+1})$ whose sign depends both on the behavior of the fluid density θ and the values of the parameters β and σ. It was shown that when $m_{\rm eff} < 0$, there were regular solutions with finite masses and sizes. In some sense the solutions presented above are a cross between the interacting scalar field solutions of [33] and the interacting real scalar field solutions of [14]. In the interior region, $0 < \xi < \xi_1$, we have a real scalar field and a fluid which interacts with the scalar field. Because of the chameleonlike behavior of the scalar field with respect to the fluid (from Fig. 3, one can see that in the interior region the field mimics the fluid) one might think of the 2 degrees of freedom associated with the scalar field and the fluid as being equivalent to the 2 degrees of freedom of a complex scalar field. While there is certainly some validity in this analogy (both the present solutions and those in [33] are nonsingular), one must avoid pushing the analogy too far since the fluid vanishes exactly at some point (i.e. $\theta = 0$ at $\xi = \xi_1$) while the complex scalar field of [33] and the real scalar of the present solution only go to zero asymptotically. Thus, after reaching the point $\xi = \xi_1$ our solutions become those of [14].

Our results are interesting since they show that by adding such a fluid one can find regular solutions, thus evading some gravitational version of Derrick's theorem [23,24] which prohibits regular solutions for the system of gravity plus a normal scalar field if the potential $V(\varphi) > 0$. Thus, the addition of the fluid was crucial to the existence of these solutions. The original, nongravitational version of Derrick's theorem [21,22] assumes not merely the regularity of solutions, but also their stability. In this paper, we have performed a preliminary stability analysis based on energy considerations. In this case, we compared the total energy of the system (including the internal and gravitational energies) and the rest energy of the gas particles. For the values of the polytropic index n and the parameter σ, and the specific coupling function f in the form of (22), this energy approach showed that the solutions studied in this paper are unstable. This was because the binding energy of the system, which was equal to the difference of the rest energy and the total energy, was negative. But, as it was shown in [34], even a positive binding energy does not guarantee stability of a system allowing its transition to an energetically more advantageous state with the same polytropic index n, but having another parameter σ. Obviously, such a process is accompanied by an ejection of the excess of energy. In our case, the possibility is not excluded that there exist regular solutions with positive binding energy, for other values of n and σ and other parameters of the scalar field β, Λ, ϕ_0. There is an additional possibility for finding stable solutions which was described at the end of Sec. IV C—since the solutions and their stability are sensitive to the form of the scalar field-fluid coupling (which in the present work had the form (22)) one could look for stable solutions by changing the form of the scalar field-fluid coupling. For example, one could try nonpolynomial couplings of the form $f(\phi) = e^{\phi}$. Finally, the energy approach to stability which was used in this paper can and should be supplemented by a study of dynamical stability along the lines of the linear stability analysis performed in [14,18,20], or by using the catastrophe theory method suggested in [35]. In future work, we plan to perform a stability analysis of the chameleon star solutions investigated here using both these approaches.

The spherically symmetric solution that we found for the system considered in this paper were called chameleon stars in analogy with chameleon cosmological models [10] because the characteristics of the scalar field (e.g. its mass) strongly depend the other fields and fluids in its environment. We characterized the behavior of these chameleon star configurations for both relativistic and nonrelativistic cases. For the relativistic chameleon stars, we found that in general the mass-energy density tended to be more concentrated toward the center of the star at $\xi = 0$. Despite this greater concentration of mass near the center of the chameleon star, the total mass of the chameleon star was lower than the corresponding relativistic stars without a scalar field. This was the result of the outer regions of the chameleon star being much less dense than a corresponding nonchameleon, relativistic star. Similar comments apply to the nonrelativistic chameleon stars. For the nonrelativistic case, we were able to find an analytical solution for the case when the scalar field was massless. This analytical solution had similar behavior to the numerically obtained solutions from Sec. IV B.

We briefly give some speculations about possible physical applications of the present solutions. Scalar fields are thought to play a role in cosmological dynamics (dark energy) and in dynamics at the galactic scale (dark matter). The present proposal is that a scalar field could play some role in astrophysics at the stellar scale in the formation of the chameleon star configurations discussed here. The fluid in our model would be provided by the star. For example, one could consider neutron stars which contain a significant amount of the scalar field. From the discussion above, our chameleon stars—both relativistic and nonrelativistic—would be less massive for a given radius despite having a higher concentration of mass near the center. Thus, one might look for neutron stars which have a larger radius for a given mass than would be expected for a normal neutron star. In this sense, chameleon stars would have the opposite behavior from hypothetical quark stars which have a larger mass for a given radius. Another possibility is that some living stars (i.e. stars which are

still fusing elements and are on the main sequence) might have trapped some significant amount of scalar field in their interior. Such living, chameleon stars would tend to have a larger density near their center, thus increasing the rate at which they fused elements. The consequence of this is that such stars would be hotter and live a shorter than expected lifespan for their mass. Again, a given chameleon star would have less overall mass than a nonchameleon star but would be nevertheless have a higher interior temperature. The physical applications of the chameleon star model suggested above would of course require the existence of stable solutions. A search for such stable solutions is a goal of future studies.

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APPENDIX: DERIVATION OF THE ENERGY-MOMENTUM TENSOR

In the case of the isentropic quasipotential flow, the Lagrangian of the continuous medium has the form [29,30]

$$L_m = p = \mathcal{W} - \varepsilon, \tag{A1}$$

where $\mathcal{W} = (\varepsilon + p)$ is the heat function, and ε is the energy density. Introducing the quasipotential

$$cS_i = c\frac{\partial S}{\partial x^i} = \mathcal{W}u_i,$$

where c is the velocity of light and S is the action for the matter, we have

$$S_iS^i = \frac{w^2}{c^2}u_iu^i = \frac{\mathcal{W}^2}{c^2},$$

from which it follows

$$\mathcal{W} = c\sqrt{g^{ik}S_iS_k}.$$

The Lagrangian (A1) now takes the form

$$L_m = c\sqrt{g^{ik}S_iS_k} - \varepsilon. \tag{A2}$$

The total action (including gravity plus all matter sources) in curvilinear coordinates is [36]

$$S = \frac{1}{c}\int L\sqrt{-g}d^4x, \tag{A3}$$

where the Lagrange density, L, above is that given in Eq. (1). By varying the action (A3) with respect to the metric, g_{ik}, one can obtain the Einstein equations and the energy-momentum tensor. In this way, the energy-momentum tensor is given by the expression

$$\frac{1}{2}\sqrt{-g}T_{ik} = \frac{\partial\sqrt{-g}\mathfrak{L}}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial\sqrt{-g}\mathfrak{L}}{\partial\frac{\partial g^{ik}}{\partial x^l}}, \tag{A4}$$

where the Lagrangian, $\mathfrak{L}$, contains only the matter components of the Lagrangian (1), namely

$$\mathfrak{L} = \frac{1}{2}g^{ik}\partial_i\varphi\partial_k\varphi - V(\varphi) + f(\varphi)L_m.$$

Since $\sqrt{-g}\mathfrak{L}$ does not depend on $\frac{\partial g^{ik}}{\partial x^l}$, the last term in (A4) equals zero. Then we have

$$\frac{1}{2}\sqrt{-g}T_{ik} = \frac{\partial\sqrt{-g}\mathfrak{L}}{\partial g^{ik}} = \mathfrak{L}\frac{\partial\sqrt{-g}}{\partial g^{ik}} + \sqrt{-g}\frac{\partial\mathfrak{L}}{\partial g^{ik}}. \tag{A5}$$

Taking into account that $dg = gg^{ik}dg_{ik} = -gg_{ik}dg^{ik}$ and also using the expressions (A1) and (A2), we find

$$\mathfrak{L}\frac{\partial\sqrt{-g}}{\partial g^{ik}} = -\frac{1}{2}\sqrt{-g}g_{ik}\mathfrak{L},$$

and

$$\sqrt{-g}\frac{\partial\mathfrak{L}}{\partial g^{ik}} = \sqrt{-g}\left(\frac{1}{2}\frac{cS_iS_k}{\sqrt{g^{ik}S_iS_k}}f + \frac{1}{2}\partial_i\varphi\partial_k\varphi\right)$$
$$= \frac{1}{2}\sqrt{-g}(f\mathcal{W}u_iu_k + \partial_i\varphi\partial_k\varphi).$$

Substituting these expressions into (A5), we finally have

$$T_{ik} = f[(\varepsilon + p)u_iu_k - g_{ik}p] + \partial_i\varphi\partial_k\varphi - g_{ik}\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\right]. \tag{A6}$$

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