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**Vladimir Dzhunushaliev**

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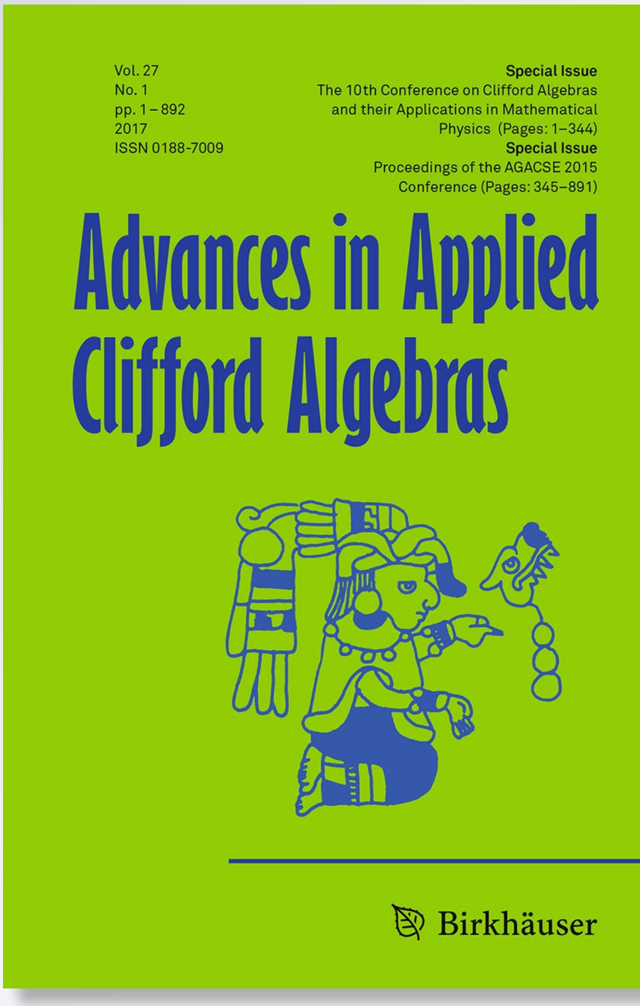


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**Advances in Applied Clifford Algebras**

# Nonassociative Generalization of Supersymmetry

Vladimir Dzhunushaliev

*To my wife Nina and childrens: Danil and Natalia*

**Abstract.** A nonassociative generalization of supersymmetry is studied, where supersymmetry generators are considered to be the nonassociative ones. Associators for the product of three and four multipliers are defined. Using a special choice of the parameters, it is shown that the associator of the product of four supersymmetry generators is connected with the angular momentum operator. The connection of operator decomposition to the hidden variables theory and alternative quantum mechanics is discussed.

**Mathematics Subject Classification.** Primary 17A70; Secondary 17D99.

**Keywords.** Supersymmetry, non-associativity.

## 1. Introduction

Supersymmetry is one of the important parts of the paradigm of modern physics. Nonassociativity is a rare guest in modern theoretical physics. Nevertheless, Ref. [1] offers many examples in the use of nonassociativity in physics. At the present time, the following nonassociative topics are under consideration: the classification of quaternionic and octonionic realizations of Clifford algebras and spinors [2]; nonassociative octonionic ternary gauge field theories based on a ternary bracket [3]; octonionic electrodynamics and the Dirac equation [4,5]; gauge theory on nonassociative spaces [6]; nonassociative geometry and a discrete spacetime [7]; the Standard model within nonassociative geometry [8].

Here we would like to show that the supersymmetry has a nonassociative generalization; to give exact definitions of three and four associators; to show that for a special definition of a four associator, this associator will be connected with the angular momentum operator; and to discuss some interesting problems arising from the obtained decomposition of quantum operators. In this connection, it is necessary to note a four-dimensional nontrivial extension of the Poincaré algebra, distinct from the supersymmetry—the so-called fractional supersymmetry [9,10]. In supersymmetry, the extensions of the Poincaré algebra are obtained from a square root of the translations, $QQ \sim P$. In cubic supersymmetry, new algebras occur as a result of using of associator $[Q, Q, Q] \sim P$, see Ref. [11].

Another interesting approach to extend supersymmetry is a ternary algebra. Ternary algebras may give a unified description of Lie algebras and superalgebras [12]. In Ref. [13] the nonassociative and noncommutative octonionic ternary gauge eld theory based on a ternary-bracket structure involving the octonion algebra is considered. The ternary bracket obeys the Nambu fundamental identity and was developed by Yamazaki [14].

## 2. The Simplest Supersymmetry Algebra

The simplest supersymmetry algebra is defined as

$$\{Q_a, Q_{\dot a}\} = Q_a Q_{\dot a} + Q_{\dot a} Q_a = 2\sigma^\mu_{a\dot a} P_\mu, \tag{2.1}$$

$$\{Q_a, Q_b\} = \left\{Q_{\dot a}, Q_{\dot b}\right\} = 0, \tag{2.2}$$

$$[Q_a, P_\mu] = [Q_{\dot a}, P_\mu] = 0, \tag{2.3}$$

$$[P_\mu, P_\nu] = 0, \tag{2.4}$$

where $Q_a, Q_{\dot a}$ are supersymmetry generators; $P_\mu = -i\partial_\mu$; $\mu = 0, 1, 2, 3$; $a = 1, 2$; $\dot a = \dot 1, \dot 2$, and the Pauli matrices $\sigma^\mu_{a\dot a}, \sigma^{a\dot a}_\mu$ are

$$\sigma^\mu_{a\dot a} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{2.5}$$

$$\sigma^{a\dot a}_\mu = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \imath \\ -\imath & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \tag{2.6}$$

with the following relations,

$$\sigma^{a\dot a}_\mu \sigma^\nu_{a\dot a} = 2\delta^\nu_\mu, \quad \sigma^{a\dot a}_\mu \sigma^\mu_{b\dot b} = 2\delta^a_b \delta^{\dot a}_{\dot b}, \quad \imath^2 = -1. \tag{2.7}$$

An inverse relation for (2.1) is

$$P_\mu = \frac{1}{4}\sigma^{a\dot a}_\mu \left\{Q_a, Q_{\dot a}\right\}. \tag{2.8}$$

The main goal of this Letter is to show that one can generalize supersymmetry in such a way that supergenerators $Q_a, Q_{\dot a}$ become nonassociative ones.

## 3. Nonassociative Generalization of the Simplest Supersymmetry Algebra

Let us define an associator as follows:

$$[x, y, z] = (xy)\, z - x\, (yz)\,. \tag{3.1}$$

Let us now define associators for supersymmetry generators. First, we define an associator for the product of three generators,

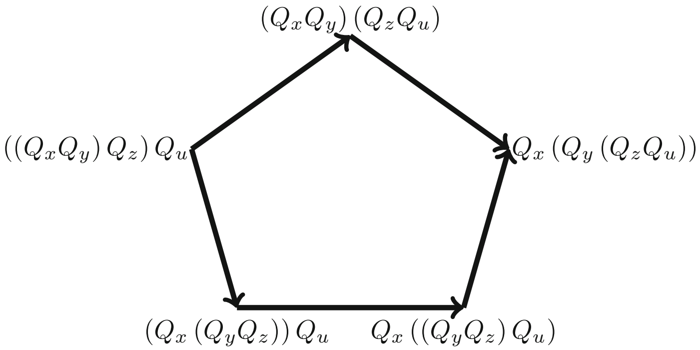
$$[Q_x, Q_y, Q_z] = 0, \tag{3.2}$$

where the triple $x, y, z$ is any combination of dotted and undotted indices $a, \dot{a}$. We define an associator for the product of four generators as

$$[Q_a, Q_b, (Q_x Q_y)] = \left[Q_{\dot{a}}, Q_{\dot{b}}, (Q_x Q_y)\right] = 0, \tag{3.3}$$
$$\left[Q_a, Q_{\dot{a}}, \left(Q_b Q_{\dot{b}}\right)\right] = \alpha_{a\dot{a}} \left\{Q_b, Q_{\dot{b}}\right\} + \beta_{b\dot{b}} \left\{Q_a, Q_{\dot{a}}\right\}, \tag{3.4}$$
$$\left[Q_{\dot{a}}, Q_a, \left(Q_b Q_{\dot{b}}\right)\right] = \gamma_{a\dot{a}} \left\{Q_b, Q_{\dot{b}}\right\} + \delta_{b\dot{b}} \left\{Q_a, Q_{\dot{a}}\right\}, \tag{3.5}$$
$$\left[(Q_a Q_b), Q_x, Q_y\right] = \left[\left(Q_{\dot{a}} Q_{\dot{b}}\right), Q_x, Q_y\right] = 0, \tag{3.6}$$
$$\left[(Q_a Q_{\dot{a}}), Q_b, Q_{\dot{b}}\right] = \tilde{\alpha}_{a\dot{a}} \left\{Q_b, Q_{\dot{b}}\right\} + \tilde{\beta}_{b\dot{b}} \left\{Q_a, Q_{\dot{a}}\right\}, \tag{3.7}$$
$$\left[(Q_a Q_{\dot{a}}), Q_{\dot{b}}, Q_b\right] = \tilde{\gamma}_{a\dot{a}} \left\{Q_b, Q_{\dot{b}}\right\} + \tilde{\delta}_{b\dot{b}} \left\{Q_a, Q_{\dot{a}}\right\}, \tag{3.8}$$
$$\left[Q_a, (Q_b Q_x), Q_y\right] = \left[Q_{\dot{a}}, \left(Q_{\dot{b}} Q_x\right), Q_y\right] = 0, \tag{3.9}$$
$$\left[Q_a, \left(Q_{\dot{a}} Q_{\dot{b}}\right), Q_b\right] = \left[Q_{\dot{a}}, (Q_a Q_b), Q_{\dot{b}}\right] = 0, \tag{3.10}$$
$$\left[Q_a, (Q_{\dot{a}} Q_b), Q_{\dot{b}}\right] = \tilde{\tilde{\alpha}}_{a\dot{a}} \left\{Q_b, Q_{\dot{b}}\right\} + \tilde{\tilde{\beta}}_{b\dot{b}} \left\{Q_a, Q_{\dot{a}}\right\}, \tag{3.11}$$

where the pair $x, y$ is any combination of dotted and undotted indices $a, \dot{a}$; $\alpha_{a\dot{a}}$ and $\beta_{b\dot{b}}$ are complex numbers.

Using pentagon identity




(3.12)

one can bind $\alpha_{a\dot{a}}, \tilde{\alpha}_{a\dot{a}}, \tilde{\tilde{\alpha}}_{a\dot{a}}$ from (3.4), (3.7), and (3.11) (where $x, y, z, u$ are any combinations of dotted and undotted indices $a, \dot{a}$). For $\beta_{a\dot{a}}, \tilde{\beta}_{a\dot{a}}, \tilde{\tilde{\beta}}_{a\dot{a}}$, we do the same thing:

$$\tilde{\tilde{\alpha}}_{a\dot{a}} = \tilde{\alpha}_{a\dot{a}} - \alpha_{a\dot{a}}, \tag{3.13}$$
$$\tilde{\tilde{\beta}}_{a\dot{a}} = \tilde{\beta}_{a\dot{a}} - \beta_{a\dot{a}}. \tag{3.14}$$

Now we would like to calculate the graded Jacobiator (ternutator):

$$\begin{aligned} J &= \left[Q_a, Q_{\dot{a}}, \left(Q_b Q_{\dot{b}}\right)\right] - \left[Q_{\dot{a}}, \left(Q_b Q_{\dot{b}}\right), Q_a\right] + \left[\left(Q_b Q_{\dot{b}}\right), Q_a, Q_{\dot{a}}\right] \\ &= \left(\beta_{b\dot{b}} - \gamma_{b\dot{b}} + \tilde{\alpha}_{b\dot{b}}\right) \left\{Q_a, Q_{\dot{a}}\right\} + \left(\alpha_{a\dot{a}} - \delta_{\dot{a}a} + \tilde{\beta}_{a\dot{a}}\right) \left\{Q_b, Q_{\dot{b}}\right\}. \end{aligned} \tag{3.15}$$

One can see that the Jacobiator will be equal to zero if we choose the following parameters:

$$\beta_{b\dot{b}} - \gamma_{b\dot{b}} + \tilde{\alpha}_{b\dot{b}} = 0, \tag{3.16}$$
$$\alpha_{a\dot{a}} - \delta_{\dot{a}a} + \tilde{\beta}_{a\dot{a}} = 0 \tag{3.17}$$

but for another choice of these parameters the Jacobian will be nonvanishing.

Now we have to check the consistency of the commutators (2.3) and (2.4). The consistency of the commutator (2.3) follows from the associator (3.2). To check the consistency of the commutator (2.4), we substitute $P_\mu$ from (2.8) into (2.4):

$$\begin{aligned}[P_\mu, P_\nu] &= \frac{1}{16}\sigma^{a\dot a}_\mu \sigma^{b\dot b}_\nu \left[\{Q_a, Q_{\dot a}\}, \{Q_b, Q_{\dot b}\}\right] \\ &= \frac{1}{16}\sigma^{a\dot a}_\mu \Big(\big[(Q_a Q_{\dot a}), (Q_b Q_{\dot b})\big] + \big[(Q_{\dot a} Q_a), (Q_b Q_{\dot b})\big] \\ &\quad + \big[(Q_a Q_{\dot a}), (Q_{\dot b} Q_b)\big] + \big[(Q_{\dot a} Q_a), (Q_{\dot b} Q_b)\big]\Big). \qquad (3.18)\end{aligned}$$

In order to evaluate the right-hand side of (3.18), we have to calculate the commutator

$$\begin{aligned}\big[(Q_a Q_{\dot a}), (Q_b Q_{\dot b})\big] = {}& \left(-\alpha_{b\dot b} + \beta_{b\dot b}\right)\{Q_a, Q_{\dot a}\} \\ & + \left(\alpha_{a\dot a} - \beta_{a\dot a}\right)\{Q_b, Q_{\dot b}\} \\ & - 2\sigma^\alpha_{a\dot b} Q_b \left(P_\alpha Q_{\dot a}\right) + 2\sigma^\alpha_{b\dot a} Q_a \left(P_\alpha Q_{\dot b}\right). \qquad (3.19)\end{aligned}$$

In order to evaluate the right-hand side of (3.19), we have employed the following chain:

$$\begin{aligned}(Q_a Q_{\dot a})(Q_b Q_{\dot b}) &\to Q_a\left(Q_{\dot a}(Q_b Q_{\dot b})\right) \to Q_a\left((Q_{\dot a} Q_b) Q_{\dot b}\right) \\ &\to Q_a\left((Q_b Q_{\dot a}) Q_{\dot b}\right) \to Q_a\left(Q_b (Q_{\dot a} Q_{\dot b})\right) \\ &\to (Q_a Q_b)(Q_{\dot a} Q_{\dot b}) \to (Q_b Q_a)(Q_{\dot a} Q_{\dot b}) \\ &\to Q_b\left(Q_a (Q_{\dot a} Q_{\dot b})\right) \to Q_b\left(Q_a (Q_{\dot b} Q_{\dot a})\right) \\ &\to Q_b\left((Q_a Q_{\dot b}) Q_{\dot a}\right) \to Q_b\left((Q_{\dot b} Q_a) Q_{\dot a}\right) \\ &\to Q_b\left(Q_{\dot b}(Q_a Q_{\dot a})\right) \to (Q_b Q_{\dot b})(Q_a Q_{\dot a}). \qquad (3.20)\end{aligned}$$

Other commutators can be found by using (3.19):

$$\begin{aligned}\big[(Q_{\dot a} Q_a), (Q_b Q_{\dot b})\big] = {}& \left(\alpha_{b\dot b} - \beta_{b\dot b}\right)\{Q_a, Q_{\dot a}\} + \left(-\alpha_{a\dot a} + \beta_{a\dot a}\right)\{Q_b, Q_{\dot b}\} \\ & + 2\sigma^\alpha_{a\dot b} Q_b \left(P_\alpha Q_{\dot a}\right) - 2\sigma^\alpha_{b\dot a} Q_a \left(P_\alpha Q_{\dot b}\right) + 2\sigma^\alpha_{a\dot a}\left[P_\alpha, Q_b Q_{\dot b}\right]\end{aligned} \qquad (3.21)$$

$$\begin{aligned}\big[(Q_a Q_{\dot a}), (Q_{\dot b} Q_b)\big] = {}& \left(\alpha_{b\dot b} - \beta_{b\dot b}\right)\{Q_a, Q_{\dot a}\} + \left(-\alpha_{a\dot a} + \beta_{a\dot a}\right)\{Q_b, Q_{\dot b}\} \\ & + 2\sigma^\alpha_{a\dot b} Q_b \left(P_\alpha Q_{\dot a}\right) - 2\sigma^\alpha_{b\dot a} Q_a \left(P_\alpha Q_{\dot b}\right) + 2\sigma^\alpha_{b\dot b}\left[Q_a Q_{\dot a}, P_\alpha\right]\end{aligned} \qquad (3.22)$$

$$\begin{aligned}\big[(Q_{\dot a} Q_a), (Q_{\dot b} Q_b)\big] = {}& \left(-\alpha_{b\dot b} + \beta_{b\dot b}\right)\{Q_a, Q_{\dot a}\} + \left(\alpha_{a\dot a} - \beta_{a\dot a}\right)\{Q_b, Q_{\dot b}\} \\ & - 2\sigma^\alpha_{a\dot b} Q_b \left(P_\alpha Q_{\dot a}\right) + 2\sigma^\alpha_{b\dot a} Q_a \left(P_\alpha Q_{\dot b}\right) \\ & - 2\sigma^\alpha_{a\dot a}\left[P_\alpha \left(Q_b Q_{\dot b}\right) + \left(Q_{\dot b} Q_b\right) P_\alpha\right] \\ & - 2\sigma^\alpha_{b\dot b}\left[(Q_a Q_{\dot a}) P_\alpha + P_\alpha (Q_{\dot a} Q_a)\right] + 4\sigma^\alpha_{a\dot a}\sigma^\beta_{b\dot b}\{P_\alpha, P_\beta\}.\end{aligned} \qquad (3.23)$$

The right-hand side of (3.21)–(3.23) are calculated by analogy to (3.20). Substituting (3.19)–(3.23) into the right-hand side of (3.18), we obtain the identity

$$[P_\mu, P_\nu] \equiv [P_\mu, P_\nu], \tag{3.24}$$

that proves the consistency of the commutator (2.4).

The numbers $\alpha_{a\dot{a}}$ and $\beta_{a\dot{a}}$ are still uncertain. Probably the most interesting case is the following:

$$\alpha_{a\dot{a}} = \zeta \sigma^\mu_{a\dot{a}} x_\mu, \tag{3.25}$$

$$\beta_{a\dot{a}} = -\zeta \sigma^\mu_{a\dot{a}} x_\mu \tag{3.26}$$

where $\zeta$ is some constant. In this case the associator $\left[Q_a, Q_{\dot{a}}, \left(Q_b Q_{\dot{b}}\right)\right]$ becomes

$$\left[Q_a, Q_{\dot{a}}, \left(Q_b Q_{\dot{b}}\right)\right] = 2\zeta \sigma^\mu_{a\dot{a}} \sigma^\nu_{b\dot{b}} M_{\mu\nu}, \tag{3.27}$$

where the coefficient $\zeta$ equalize the dimensions of the right and left hand sides of (3.27). The operator

$$M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu \tag{3.28}$$

is the angular momentum operator. The inverse relation for (3.27) is

$$M_{\mu\nu} = \frac{1}{8\zeta} \sigma^{a\dot{a}}_\mu \sigma^{b\dot{b}}_\nu \left[Q_a, Q_{\dot{a}}, \left(Q_b Q_{\dot{b}}\right)\right]. \tag{3.29}$$

Let us check the Nambu fundamental identity

$$\begin{aligned}\left[Q_x, Q_y, \left[Q_a, Q_{\dot{a}}, \left(Q_b Q_{\dot{b}}\right)\right]\right] = & \left[\left[Q_x, Q_y, Q_a\right], Q_{\dot{a}}, \left(Q_b Q_{\dot{b}}\right)\right] \\ & + \left[Q_a, \left[Q_x, Q_y, Q_{\dot{a}}\right], \left(Q_b Q_{\dot{b}}\right)\right] \\ & + \left[Q_a, Q_{\dot{a}}, \left[Q_x, Q_y, \left(Q_b Q_{\dot{b}}\right)\right]\right]. \end{aligned} \tag{3.30}$$

In the consequence of (2.1), (3.2), (3.4) and (3.5) the Nambu fundamental identity (3.30) has the following form

$$\begin{aligned}&\alpha_{a\dot{a}} \sigma^\mu_{b\dot{b}} \left[Q_x, Q_y, P_\mu\right] + \beta_{b\dot{b}} \sigma^\mu_{a\dot{a}} \left[Q_x, Q_y, P_\mu\right] \\ &= \begin{cases} 0, \quad \text{if } x, y \text{ are both either dotted or undotted indices;} \\ \alpha_{c\dot{c}} \sigma^\mu_{b\dot{b}} \left[Q_a, Q_{\dot{a}}, P_\mu\right] + \beta_{b\dot{b}} \sigma^\mu_{c\dot{c}} \left[Q_a, Q_{\dot{a}}, P_\mu\right], \quad \text{if } x = c, y = \dot{c}; \\ \gamma_{c\dot{c}} \sigma^\mu_{b\dot{b}} \left[Q_a, Q_{\dot{a}}, P_\mu\right] + \delta_{b\dot{b}} \sigma^\mu_{c\dot{c}} \left[Q_a, Q_{\dot{a}}, P_\mu\right], \quad \text{if } x = \dot{c}, y = c. \end{cases}\end{aligned} \tag{3.31}$$

Thus the Nambu fundamental identity is trivial one $0 = 0$ since $[Q_x, Q_y, P_\mu] = 0$, $P_\mu$ is the associative operator.

Let us note that one can generalize the expression (3.27) in the following way[1]

$$\begin{aligned}\left[Q_a, Q_{\dot{a}}, \left(Q_b Q_{\dot{b}}\right)\right] = & a_1 \epsilon_{ab} \epsilon_{\dot{a}\dot{b}} \mathbb{I} + \sigma^\mu_{a\dot{a}} \sigma^\nu_{b\dot{b}} \left(a_2 P_\mu P_\nu + a_3 M_{\mu\nu}\right) \\ & + a_4 \eta_{\mu\nu} \sigma^\mu_{a\dot{a}} \sigma^\nu_{b\dot{b}} \mathbb{I} + a_5 \sigma^{\mu\nu}_{ab} \sigma^{\rho\tau}_{\dot{a}\dot{b}} M_{\mu\nu} M_{\rho\tau} + \cdots \end{aligned} \tag{3.32}$$

where $a_{1,2,\ldots,5}$ are numerical constants; $\sigma^{\mu\nu}_{ab} = [\sigma^\mu, \sigma^\nu]_{ab}$, $\sigma^{\mu\nu}_{\dot{a}\dot{b}} = [\sigma^\mu, \sigma^\nu]_{\dot{a}\dot{b}}$, and $\epsilon_{ab}, \epsilon_{\dot{a}\dot{b}}$ are the antisymmetric symbols.

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[1] Thanks for the referee for this comment.

## 4. Decomposition of Quantum Mechanics Operators

As we saw above, the momentum and the angular momentum operators in quantum mechanics can be decomposed as a multilineal combination of constituents $Q_a, Q_{\dot{a}}$,

$$P_\mu = \frac{1}{4}\sigma_\mu^{a\dot{a}} \left\{Q_a, Q_{\dot{a}}\right\}, \tag{4.1}$$

$$M_{\mu\nu} = \frac{1}{8\zeta}\sigma_\mu^{a\dot{a}}\sigma_\nu^{b\dot{b}} \left[Q_a, Q_{\dot{a}}, \left(Q_b Q_{\dot{b}}\right)\right]. \tag{4.2}$$

It was shown in Ref. [15] that the nonrelativistic spin operator $s_i$ can be decomposed as the commutator of octonions (for the definition of octonions and other details, see Appendix A),

$$s_i = -\frac{1}{4}\epsilon_{ijk}\left[q_{j+3}, q_{k+3}\right], \quad i, j, k = 1, 2, 3, \tag{4.3}$$

where $q_i$ are split-octonions.

All this shows that at least some operators in quantum mechanics can be decomposed as a multilineal combination of constituents.

The decompositions (4.1)–(4.3) lead to another interesting question: whether there is a nonassociative algebra $\mathcal{A}$ in which there exists an associative algebra $\mathcal{Q} \subset \mathcal{A}$ such that $\mathcal{Q}$ is an algebra of quantum operators?

## 5. Discussion and Conclusions

In this Letter we have shown that the idea of supersymmetry can be extended with the inclusion of nonassociativity into supersymmetry. We have defined associators with three and four multipliers and have shown the consistency of these definitions. It is shown that for some special choice of the parameters of four associator such an associator gives rise to an angular momentum operator.

We have seen that momentum, angular momentum, and spin operators have nonassociative decompositions. This allows us to ask the question whether there is a nonassociative algebra $\mathcal{A}$ in which there exists an associative algebra $\mathcal{Q} \subset \mathcal{A}$ such that $\mathcal{G}$ is the algebra of quantum operators. This means that the operators $G \in \mathcal{G}$ are the operators of either quantum mechanics or quantum field theory. If the answer is positive, then an interesting situation occurs: any quantum operator can be decomposed as a multilineal combination of nonassociative constituents. In such a situation one can give a positive answer to the old question whether there is an alternative quantum mechanics. The fact of the matter is that one can rearrange brackets in the nonassociative operator decomposition of quantum operator and redefine the action for the wave-function operator.

In connection with the decompositions (4.1)–(4.3), one can remember the hidden variables theory (HVT) where hidden variables are the classical ones. The HVT argues that a quantum state of a physical system does not give a complete description of the system. It is assumed in the HVT that quantum mechanics represents a statistical approximation of an unknown deterministic theory, where all observables have defined values fixed by unknown variables. The difference between the decompositions (4.1)–(4.3) presented here and the HVT is that constituents $Q_a, Q_{\dot{a}}$ are nonassociative quantities but in the HVT the unknown variables are classical ones. Moreover, one can show [16] that nonassociative quantities are unobservable ones. This means that nonassociative decompositions cannot be considered as the HVT.

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## Appendix A. Octonions

The split-octonions are nonassociative numbers. The split-octonions have the following commutators and associators:

$$[q_{i+3}, q_{j+3}] = -2\epsilon_{ijk} q_k, \tag{A.1}$$

$$[q_i, q_j] = 2\epsilon_{ijk} q_k, \tag{A.2}$$

$$(q_{i+3}, q_{j+3}, q_{k+3}) = (q_{i+3} q_{j+3})\, q_{k+3} - q_{i+3}\,(q_{j+3} q_{k+3}) = 2\epsilon_{ijk} q_7, \tag{A.3}$$

where $i, j, k = 1, 2, 3$. The commutator (A.2) shows that $q_i$ (with $i = 1, 2, 3$) form a subalgebra. This subalgebra is called the quaternion algebra $\mathbb{H}$; $q_{1,2,3}$ are quaternions. The commutator (A.2) is the same as the commutator relationship for spin operators $s_i = \sigma_i$ ($\sigma_i$ are the Pauli matrices). The relations (A.1)–(A.3) can be written in the Zorn vector matrix representation, where the octonion $o$ is written in the form

$$o = \sum_{i=1}^{8} \alpha_i q_i = \begin{pmatrix} a & \vec{x} \\ \vec{y} & b \end{pmatrix}, \tag{A.4}$$

where $q_0 = 1$; $\alpha_i$ are numbers; $a, b$ are real numbers; and $\vec{x}, \vec{y}$ are 3-vectors. The product of two octonions is defined as

$$\begin{pmatrix} a & \vec{x} \\ \vec{y} & b \end{pmatrix} \begin{pmatrix} c & \vec{u} \\ \vec{v} & d \end{pmatrix} = \begin{pmatrix} ac + \vec{x}\cdot\vec{v} & a\vec{u} + d\vec{x} - \vec{y}\times\vec{v} \\ c\vec{y} + b\vec{v} + \vec{x}\times\vec{u} & bd + \vec{y}\cdot\vec{u} \end{pmatrix} \tag{A.5}$$

here $(\cdot)$ and $[\times]$ denote the usual scalar and vector products. Let us introduce the orthogonal vectors $\vec{e}_i$, where $i = 1, 2, 3$, such that $\vec{e}_i \times \vec{e}_j = \epsilon_{ijk}\vec{e}_k$ and $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$. Then the split-octonions have the following Zorn vector matrix representation:

$$1 = \begin{pmatrix} 1 & \vec{0} \\ \vec{0} & 1 \end{pmatrix}, \quad q_7 = -\begin{pmatrix} 1 & \vec{0} \\ \vec{0} & -1 \end{pmatrix}, \quad q_i = \begin{pmatrix} 0 & -\vec{e}_i \\ \vec{e}_i & 0 \end{pmatrix},$$
$$q_{i+3} = \begin{pmatrix} 0 & \vec{e}_i \\ \vec{e}_i & 0 \end{pmatrix}, \tag{A.6}$$

where $i = 1, 2, 3$. Thus, the nonrelativistic spin operators have two matrix representations: the first one is the Pauli matrix representation with $s_i = \frac{\sigma_i}{2}$, and the second one is the Zorn vector matrix representation (A.6) with $s_i = \frac{\imath}{2} q_i, i = 1, 2, 3$.

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Vladimir Dzhunushaliev
Department of Theoretical and Nuclear Physics
KazNU
Almaty 050040
Kazakhstan

and

IETP, Al-Farabi Kazakh National University
Almaty 050040
Kazakhstan
e-mail: `v.dzhunushaliev@gmail.com`



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