Singularity-free model of electric charge in physical vacuum: Non-zero spatial extent and mass generation

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We propose a model of a spinless electrical charge as self-consistent field configuration of the electromagnetic (EM) field interacting with the physical vacuum effectively described by the logarithmic quantum Bose liquid. We show that, in contrast to the EM field propagating in a trivial vacuum, a regular solution does exist, and both its mass and spatial extent emerge naturally from dynamics. It is demonstrated that the charge and energy density have the Gaussian-like form, the solution in the logarithmic model is stable and energetically favorable, unlike the one obtained in a model with quartic potential.

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1. INTRODUCTION

Two of the oldest, yet actual problems in fundamental particle physics are the problems of finite self-energy and possible extendedness of electrically charged elementary particles. The proper development of this research direction is complicated by the fact that no experimental evidence of the internal structure and spatial extent of, e.g., the electron has been found down to the 10^{-16} cm. However, the mere postulate that a certain amount of matter with mass, charge and spin can be located inside the set of zero spatial measure looks implausible to a physicist's mind. Moreover, this assumption might be one of the reasons why the unphysical divergences appear in quantum field theory (QFT). As a matter of fact, this difficulty already arises at the classical level: according to the standard theory of electromagnetism, the electrical field of a point charge in completely empty space (trivial vacuum) is described by the inverse square law, therefore, the energy density of the electrical field integrated all over the whole space turns out to be infinite. As a result, the total mass-energy of the point charge together with its field becomes infinite under those conditions, therefore, such system would be impossible to budge. At the quantum level, this problem manifests itself in the ultraviolet divergences appearing in loop diagrams. In some theories these divergences can be removed by means of regularization and renormalization procedures, which can be very useful for doing specific computations but do not shed much light upon the essence of the problem [1]. Thus, theoretical attempts towards its better understanding should not be abandoned.

Historically the very first effort was probably the model which described the electron as a ball with spatially distributed electrical charge. That model was in conflict with relativity because the ball was assumed to be absolutely rigid, besides, the description of spin was not clear. Similar difficulties were found in the models proposed by Abraham and Lorentz [2], although work in that direction continues [3]. Among other noticeable research directions one should also mention the Dirac's shell model of the extended electron and its subsequent modifications and variations [4]. Yet another model of an extended electrical charge was the Einstein's wormhole approach, according to which the electrical flux lines enter one side of the wormhole and exit from another. Then the front side looks like a negative charge and the rear like a positive charge. The wormhole models were criticized by Wheeler for issues of stability, non-quantized charge, wrong mass-charge ratio and spin [5]. Numerous attempts towards finding regular particle-like solutions were made in conventional and nonlinear electrodynamics [6], both with and without engaging general relativity [7–11]. Another interesting approach is the wave-corpuscle mechanics, see a review [12]. The general mathematical formalism used there formally resembles the one used in this paper. The important difference, however, is about underlying physics: the origin and explicit form of their nonlinear self-interacting wave term $G(|\psi|^2)$ are not specified on physical grounds and a fully satisfactory particle-like solution is not given.

One should definitely also mention the popular approach to the classical electron model using the Einstein-Dirac [13] and Kerr-Newman (KN) solutions [14], an extensive bibliography can be found in [15]. While the original KN

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solution does have the correct gyromagnetic ratio, it also contains naked singularity and thus requires an additional mechanism to circumvent the regularization problem, and the story is far from being complete yet.

Intuitively, from a fundamental theory one would expect that spatial extent is not *ab initio* built-in but naturally emerges from dynamics. In this paper we propose a model of a charged particle whose spatial extent, observable charge and mass emerge as a result of the interaction of the EM field with the physical vacuum. For simplicity we neglect internal degrees of freedom, such as spin, isospin, *etc.*, so that we can assume the spherical symmetry where possible. It turns out that the resulting solution describes a charged object which does not have a boundary in a classical sense, its stability is supported not by surface tension but by nonlinear quantum effects in the bulk. This makes our model more realistic from the quantum-mechanical point of view - since the actual observability of a definite boundary with smooth surface would be as contradictory to the quantum uncertainty principle as the notion of a smooth trajectory or worldline in the quantum realm. This can be shown by performing a simple *Gedankenexperiment*: making an extended object with a definite boundary to propagate through space and measuring the velocity and position uncertainties on its surface. In turn, it means that the surface tension is a well-defined notion only in the classical limit, but for more fundamental purposes it must be used with utmost care.

The structure of the paper is as follows. The phenomenological approach to physical vacuum is described in the next section, the main equations of the model can be found in section 3, the regular solution and its properties are analyzed, both analytically and numerically, in section 4. A comparison between our solution and its Higgs-type (quartic) counterpart is done in section 5, and the conclusions are drawn in section 6.

2. PHYSICAL VACUUM

As mentioned above, the Coulomb divergence problem essentially means that one cannot find regular particle-like solutions of the Maxwell field in empty space, not even in general relativity [16]. However, from a quantum physicist's point of view this problem is not as severe as it looks to a non-quantum theorist, because the notion of absolutely empty space (or "mathematical vacuum") cannot be realized in Nature anyway. This is because the existence of such space seriously contradicts quantum-mechanical laws. According to the latter, the genuine (physical) vacuum must be a non-trivial quantum medium which acts as a non-removable background and affects particles propagating through [1, 17].

At this time, no commonly accepted theory of physical vacuum exists, whereas the amount of experimental and observational data is still too far from being conclusive to single out only one model. One of the candidate theories lies within the framework of the superfluid vacuum approach [18–20]. This theory is based on the idea [21, 22] that a physical vacuum can be viewed as some sort of background superfluid condensate described by the logarithmic wave equation (the latter has been studied long ago on grounds of the dilatation covariance and separability [23, 24]). It was shown that small fluctuations of the logarithmic condensate obey the Lorentz symmetry and can be interpreted not only as the relativistic particle-like states but also as the gravitational ones [22] depending on a type of mode. Thus, in this approach the Lorentz symmetry is not an exact symmetry of Nature, but rather pertinent to small fluctuations of the physical vacuum and thus gets deformed at high energies and/or momenta.

As long as this approach must be fully consistent and applicable to reality, one would naturally expect that the behaviour of the conventional Maxwell field becomes regular in presence of particle-like solutions when the empty space is replaced by the logarithmic vacuum condensate. This issue is going to be the main subject of the current study. On top of that, one would expect that the spatial extent mentioned above must appear naturally in the approach. This was already shown at the non-relativistic level in [25], so here we are going to see what happens in the relativistic case.

The effective low-energy Lagrangian for the Maxwell field interacting with the small fluctuations of the physical vacuum was proposed in [22] and the mass generation mechanism was demonstrated which was analogous to the Higgs one. In this paper we propose a different mass generation mechanism which uses the same Lagrangian except that the scalar potential is assumed to open down. As it will be shown below, the latter solves the issue of a wrong sign of the quadratic (mass) term of the potential at the energies above symmetry-breaking scale - when symmetry is unbroken and false vacuum is stable. In case of three spatial dimensions the action is proportional to $\int d^4x\, \mathcal{L}$ where, adopting the natural units $c = \hbar = \epsilon_0 = 1$ and metric signature $(+ - - -)$, we assume

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\breve{a}}{2} \left| D_\mu \Psi \right|^2 - V(|\Psi|^2), \tag{1}$$

where $D_\mu = \partial_\mu + i g A_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the vacuum-induced field potential is defined as (up to an additive constant)

$$V(|\Psi|^2) = -\breve{\beta}^{-1} \left\{ |\Psi|^2 \left[\ln\left(\breve{a}^3 |\Psi|^2\right) - 1 \right] + \breve{a}^{-3} \right\}, \tag{2}$$

where $\breve{a}$ and $\breve{\beta}$ are parameters of dimensionality length in adopted units; in the underlying theory of superfluid vacuum the parameter $\breve{\beta}$ is related to the quantum (non-thermal) kind of temperature which is conjugated to the Everett information entropy [25] whereas $\breve{a}$ could be related to the characteristic inhomogeneity scale of superfluid [26]. The potential (2) is regular in the origin - while the logarithm itself diverges there, the factor $|\Psi|^2$ recovers regularity. It has a local maximum at $|\Psi|_{\rm max} = \breve{a}^{-3/2}$, i.e., it always has the (upside-down) Mexican-hat shape if plotted as a function of Ψ, see Fig. 1. In what follows we call this potential *logarithmic* - due to the property $dV/d|\Psi|^2 \propto \ln(\breve{a}^3|\Psi|^2)$ which yields the logarithmic term in the corresponding field equation.

We emphasize that, according to the approach, the Lagrangian (1) is an approximate one thus it is not valid for arbitrary large (short) scales of energy (length) which results in the field Ψ cannot take arbitrarily large values:

$$|\Psi| \leqslant |\Psi_c| < \infty, \tag{3}$$

where $|\Psi_c| = \lim\limits_{E\to E_0} |\Psi|$ is some limit value corresponding to the cutoff energy scale E_0 which is also the characteristic energy scale of the vacuum. In the effective theory the appearance of upper bound for $|\Psi|$ will be shown in section 4.2 below whereas in a full theory it could result from, for instance, the normalization condition for Ψ, similarly to the one in the theory of Bose-Einstein condensation. The constraint (3) also means that the potential (2) does not have to be bounded from below, as in a standard relativistic QFT. Alternatively, one can take the potential (2) with an opposite sign, thus making it bounded from below at positive $\breve{\beta}$, but treat Ψ as a phantom field.

Further, performing the rescaling

$$\psi = \sqrt{\breve{a}}\Psi,\ \ \beta = \breve{a}\breve{\beta},\ \ a = \breve{a}^{2/3}, \tag{4}$$

we can rewrite (1) and (2) in a more habitual form:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left|D_\mu\psi\right|^2 - V(\psi), \tag{5}$$

$$V(\psi) = -\beta^{-1}\left\{|\psi|^2\left[\ln\left(a^3|\psi|^2\right) - 1\right] + \frac{1}{a^3}\right\}. \tag{6}$$

Notice that by expanding the potential in the vicinity of $|\psi|^2 = \varepsilon/a^3$, ε being a non-negative dimensionless number, one arrives at the following perturbative expression (up to an additive constant):

$$V(\psi) \approx \frac{\lambda_{\rm eff}}{4!}|\psi|^4 + \frac{1}{2}m_b^2|\psi|^2 + \mathcal{O}\left((|\psi|^2 - \varepsilon/a^3)^3\right), \tag{7}$$

where $\lambda_{\rm eff} = -12a^3/\varepsilon\beta$ is the effective quartic coupling and $m_b = \sqrt{2(1-\ln\varepsilon)/\beta}$. If its radicand is non-negative then the latter can be interpreted as the mass of an effective scalar particle (before the symmetry breaking). Indeed, one can always quantize the approximate model by analogy with a quartic (hence renormalizable) scalar QFT in a vicinity of the non-trivial vacuum represented by the ground-state solution of the original model (6) which will be discussed in the following sections. To date, a number of different quantization approaches have been developed for such cases - see, e.g., works [27, 28] and references therein.

Thus, we have expressed the physical parameters of the scalar sector, such as mass and coupling, in terms of the primary parameters of our theory. If the value of ε is close to one (or, at least, less than the base of natural logarithm) then it is indeed important that the potential (6) opens down (cf. $\breve{a} > 0$ and $\breve{\beta} > 0$) otherwise the quadratic term would appear with a wrong sign. Also the effective quartic coupling turns out to be negative in this case which is a remarkable difference from the standard Higgs potential and thus it can serve for experimental testing. In any case, the interpretation based on (7) is only approximate (for instance, such series expansion does not converge to (6) for very small $|\psi|$), therefore, in what follows we will be working with the exact expression for V.

3. FIELD EQUATIONS

The field equations corresponding to the Lagrangian (5) are given by

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{8}$$

$$D_\mu D^\mu\psi + \frac{\partial V}{\partial\psi^*} = \left[D_\mu D^\mu + \beta^{-1}\ln\left(a^3|\psi|^2\right)\right]\psi = 0, \tag{9}$$

where the current j^ν is defined as

$$j^\nu = ig\left[(D^\mu\psi)^*\psi - \psi^*(D^\mu\psi)\right]. \tag{10}$$

We look for the solution in the electrostatic form

$$A_\mu = \left[\phi(r),\vec{0}\right],\ \psi(t,r) = \mathrm{e}^{-iEt}\psi(r), \tag{11}$$

with E being a real-valued constant. Then the equations of motion become simply

$$\psi'' + \frac{2}{r}\psi' = -(E-g\phi)^2\psi - \beta^{-1}\psi\ln(a^3\psi^2), \tag{12}$$

$$\phi'' + \frac{2}{r}\phi' = -2g(E-g\phi)\psi^2, \tag{13}$$

where the prime means the derivative with respect to r. Introducing the quantities

$$x = \frac{r}{\sqrt{\beta}},\ \tilde{\psi} = a^{3/2}\psi = \breve{a}^{3/2}\Psi,\ \tilde{\phi} = -g\phi\sqrt{\beta} = -g\phi\sqrt{\breve{a}\breve{\beta}}, \tag{14}$$

$$\tilde{E} = E\sqrt{\beta} = E\sqrt{\breve{a}\breve{\beta}},\ \tilde{g} = g\sqrt{\frac{\beta}{a^3}} = g\sqrt{\frac{\breve{\beta}}{\breve{a}}}, \tag{15}$$

we obtain

$$\tilde{\psi}'' + \frac{2}{x}\tilde{\psi}' = -\left(\tilde{E}+\tilde{\phi}\right)^2\tilde{\psi} - \tilde{\psi}\ln(\tilde{\psi}^2), \tag{16}$$

$$\tilde{\phi}'' + \frac{2}{x}\tilde{\phi}' = 2\tilde{g}^2\left(\tilde{E}+\tilde{\phi}\right)\tilde{\psi}^2, \tag{17}$$

where the prime means the derivative with respect to x when applied to tilded quantities. One can see that equations depend on only one parameter $\tilde{g}$ whereas $\tilde{E} = \tilde{E}(\tilde{g})$ can be treated as an eigenvalue at a given $\tilde{g}$. The full theory of physical vacuum would provide the value of $\tilde{E}$ (or, equivalently, $\tilde{\psi}(0)$, see the analytical solution section below) as a function of the primary parameters (from, e.g., some sort of normalization condition, cf. [25]) but in the approximate theory $\tilde{E}$ stays a free parameter which can be fixed only from external considerations.

Further, for a given solution, the energy density ϵ is defined as

$$\epsilon = \frac{1}{2}D_0\psi^* D^0\psi + \frac{1}{2}D_i\psi^* D^i\psi + V(\psi) + \frac{1}{2}|\vec{\mathcal{E}}|^2, \tag{18}$$

For an observer in the reference frame associated with the center of mass of a localized solution the quantity W/c^2 is equivalent to the rest mast of a corresponding particle.

4. PARTICLE-LIKE SOLUTION AND ITS PROPERTIES

4.1. Asymptotic behavior

As long as we are about to search for the regular solution then the functions $\tilde{\psi}(x)$ and $\tilde{\phi}(x)$ should have the following behavior near the origin

$$\tilde{\psi}(x) = \tilde{\psi}_0 + \tilde{\psi}_2\frac{x^2}{2} + \mathcal{O}(x^4), \tag{19}$$

$$\tilde{\phi}(x) = \tilde{\phi}_0 + \tilde{\phi}_2\frac{x^2}{2} + \mathcal{O}(x^4). \tag{20}$$

When substituting this into (16) and (17) we obtain the solution

$$\tilde{\psi}_2 = -\frac{\tilde{\psi}_0}{3}\left[\left(\tilde{E}+\tilde{\phi}_0\right)^2 + \ln\tilde{\psi}_0^2\right], \tag{21}$$

$$\tilde{\phi}_2 = \frac{2}{3}\tilde{g}^2\left(\tilde{E}+\tilde{\phi}_0\right)\tilde{\psi}_0^2. \tag{22}$$

Further, the asymptotic behavior at $x \to \infty$ is given by

$$\tilde{\psi} \to \mathrm{e}^{\frac{1}{2}(3-\tilde{E}^2-x^2)} \left[1 + \mathcal{O}(\tilde{g}^2)\right], \ \tilde{\phi} \to -\frac{\tilde{q}}{x}, \tag{23}$$

where $\tilde{q}$ is some constant to be determined. It is instructive to relate the bare charge g to the observable one $q = \tilde{q}/g$. By integrating (13) we obtain

$$r^2 \mathcal{E} = 2g \int\limits_0^r (E - g\phi)\, \psi^2 r^2 dr, \tag{24}$$

and in the limit $r \to \infty$ we have $r^2\mathcal{E} \to q$ hence

$$q = 2g \int\limits_0^\infty (E - g\phi)\, \psi^2 r^2 dr. \tag{25}$$

Thus, we arrive at the following relation between the bare and observable charges

$$q = \frac{2g\beta}{a^3} I(\tilde{g}) = 2\tilde{g}\sqrt{\frac{\beta}{a^3}} I(\tilde{g}), \tag{26}$$

where we denoted

$$I(\tilde{g}) = \int\limits_0^\infty x^2 \left(\tilde{E} + \tilde{\phi}\right) \tilde{\psi}^2 dx. \tag{27}$$

The expression (23) shows us that at large distance we recover the Coulomb potential while the field $\tilde{\psi}$ decreases exponentially. Thus, the field $\tilde{\psi}$ is in fact unobservable, unless very short length scales are probed. From the asymptotics of the solution one can infer that the charge radius of the solution is determined by the parameter β:

$$\text{size} \sim \sqrt{\beta} \sim \sqrt{\breve{a}\breve{\beta}}, \tag{28}$$

which essentially means that the combination of the parameters $\beta = \breve{a}\breve{\beta}$ must have an extremely small value for the known elementary particles. For instance, if one takes the values of the classical radius e^2/m as the conservative estimates, then for the electron and muon one would obtain the constraints $\beta_{(e)} < 10^{-26}$ cm^2 and $\beta_{(\mu)} < 10^{-32}$ cm^2, although it is not entirely clear whether the classical radius should be analogous to the "smearing" size which is our definition of size here.

4.2. Approximate analytical solution

While the exact expression for a full analytical solution is unknown, it is possible to solve the system (16) and (17) in the approximation of weak EM coupling

$$\tilde{g}^2 \ll 1, \tag{29}$$

which is equivalent to $g^2 \ll a^3/\beta$ or $g^2 \ll \breve{a}/\breve{\beta}$. The observational constraints suggest that this approximation might have a good chance to be valid for the known elementary particles - unless $\breve{a}$ turns out to be very small. On the other hand, as long as our approach is an effective one it has certain applicability conditions - and one of them is that the vacuum effects predominate the electromagnetic ones. Therefore, large values of $\tilde{g}$ might push our approach outside its applicability range and thus the corresponding approximation is not very interesting from the physical point of view.

Thus, imposing the boundary conditions

$$\tilde{\phi}(0) < \infty, \ \tilde{\phi}(+\infty) = 0, \ \tilde{\psi}(+\infty) < \infty, \ \tilde{\psi}'(0) = 0, \tag{30}$$

one obtains the solution which is regular for $0 \leqslant r \leqslant +\infty$ (see appendix for the details of derivation):

$$\tilde{\phi} = -\frac{1}{2x}\sqrt{\pi}\tilde{g}^2\tilde{E}\,\mathrm{e}^{3-\tilde{E}^2}\mathrm{erf}(x) + \mathcal{O}(\tilde{g}^4), \tag{31}$$

$$\tilde{\psi} = \mathrm{e}^{\frac{1}{2}(3-\tilde{E}^2-x^2)}\left\{1 + \frac{1}{4}\tilde{g}^2\tilde{E}^2\mathrm{e}^{3-\tilde{E}^2-x^2}\left[1 + \frac{\sqrt{\pi}}{2x}\left(2x^2+1\right)\mathrm{e}^{x^2}\mathrm{erf}(x)\right]\right\} + \mathcal{O}(\tilde{g}^4), \tag{32}$$

where dimensionless energy $\tilde{E}$ can be also expressed via the boundary value

$$\tilde{E} = \sqrt{3 - \ln\left(\tilde{\psi}_0^2\right)}\left(1 - \frac{1}{2}\tilde{g}^2\tilde{\psi}_0^2\right) + \mathcal{O}(\tilde{g}^4), \tag{33}$$

with the square root being defined up to a sign. Of course, this formula is valid only if a magnitude of $\tilde{\psi}$ is bounded from above:

$$|\tilde{\psi}| \leqslant |\tilde{\psi}_0| \leqslant \mathrm{e}^{3/2}, \tag{34}$$

which *a posteriori* affirms the condition of applicability (3), although this upper bound does not necessarily saturate the critical value there: $|\Psi_c| \geqslant (\mathrm{e}/\breve{a})^{3/2}$.

Further, one can check that the electric part has the Coulomb behaviour at large r indeed, then the effective charge can be computed as

$$\tilde{q} = \frac{1}{2}\sqrt{\pi}\tilde{g}^2\tilde{E}\,\mathrm{e}^{3-\tilde{E}^2} + \mathcal{O}(\tilde{g}^4), \tag{35}$$

therefore, the observable charge,

$$q = \tilde{q}/g \approx \frac{1}{2a^{3/2}}\sqrt{\pi\beta}\tilde{E}\,\mathrm{e}^{3-\tilde{E}^2}\tilde{g} \approx \frac{1}{2\breve{a}}\sqrt{\pi\breve{\beta}}\tilde{E}\,\mathrm{e}^{3-\tilde{E}^2}g, \tag{36}$$

depends on the whole combination of parameters describing the interaction of the electromagnetic field with a physical vacuum. The dimensionless total energy (**??**) turns out to be

$$\tilde{W} = \frac{3}{16}\sqrt{\pi}\,\mathrm{e}^{3-\tilde{E}^2}\left[2\tilde{E}^2 - 1 + \frac{\tilde{g}^2}{3\sqrt{2}}\tilde{E}^2\left(12\tilde{E}^2 - 13\right)\mathrm{e}^{3-\tilde{E}^2}\right] + \mathcal{O}(\tilde{g}^4), \tag{37}$$

which can be written also in terms of the observable charge q and rest mass W:

$$W \approx W_{(0)}\left[1 + 4\sqrt{2}\beta^{-1}a^3q^2\mathrm{e}^{\tilde{E}^2-3}\,\frac{\tilde{E}^2-13/12}{\tilde{E}^2-1/2}\right], \tag{38}$$

where

$$W_{(0)} = \frac{3}{2}\frac{\pi^{3/2}\sqrt{\beta}}{a^3}\left(\tilde{E}^2 - \frac{1}{2}\right)\mathrm{e}^{3-\tilde{E}^2}, \tag{39}$$

so one can see that the obtained formula does not contain any divergences.

One can notice also that mass W does not vanish when charge is set to zero which indicates that the theory is also capable of incorporating non-charged particles into the scheme, by taking the corresponding limit. In fact, the mass formula (38) implies that for an electrically charged particle with mass W there can exist not only an antiparticle of the same mass but also a neutral particle of the related mass $W_{(0)}$. It is interesting that the ratio $W/W_{(0)}$ grows exponentially with growing $|\tilde{E}|$ which results in two possible scenarios: (i) the mass of a neutral partner is very small (yet non-zero) as compared to the mass of a charged one: this happens if $|\tilde{E}| \gg 1$ (or, equivalently, $|\tilde{\psi}_0| \ll 1$), (ii) if $|\tilde{E}|$ is of order one or less then both masses would be of the same order of magnitude. The possible phenomenological implications of this mechanism are discussed in the conclusion.

4.3. Numerical solution and stability

While we have managed to find the approximate regular solution analytically, it is important to check that a regular solution exists for non-small $\tilde{g}$'s and terms with higher-order powers of $\tilde{g}$ would not introduce any spatial singularities.

For this purpose we are going to solve the equations (16) and (17) numerically. For the computations we choose $\tilde{g} = 1$ and the following boundary conditions

$$\tilde{\psi}(0) = 0.1, \quad \tilde{\psi}'(0) = 0, \quad \tilde{\phi}(0) = -0.1, \quad \tilde{\phi}'(0) = 0, \tag{40}$$

and $\tilde{E}$ is treated as an eigenvalue. It should be noted that $\tilde{\phi}(x)$ must be always taken non-positive on the positive semi-axis of x, due to the asymptotic requirements (23). The numerical solution is presented in Fig 2, and in Fig. 3 the corresponding profiles of the electric field $\tilde{\mathcal{E}} = -d\tilde{\phi}/dx$ and $x^2\tilde{\mathcal{E}}$ are given. From these one see that the electric field is regular at the origin and asymptotically has the Coulomb behavior. The profile of the dimensionless energy density is shown in Fig. 4.

The direct stability analysis of the solution is complicated by the fact that the perturbed electric field becomes time-dependent, which leads to the appearance of a magnetic field such that this system cannot be regarded as spherically symmetric anymore. However, it is still possible to use the energy-based arguments as well as to investigate the behavior of an effective Schroedinger equation potential. Let us consider the dimensionless total energy $\tilde{W}$ given by (**??**) as well as the energy of the field $\tilde{\psi}$ alone, $\tilde{W}_\psi = \tilde{W}|_{\phi\to 0}$, and the energy of the electric field alone, $\tilde{W}_\phi = \tilde{W}|_{\psi\to 0}$. Then we can define the dimensionless binding energy as

$$\Delta\tilde{W} = \tilde{W} - (\tilde{W}_\phi + \tilde{W}_\psi) = \frac{1}{2}\int\limits_0^\infty \tilde{\phi}\left(\tilde{\phi} + 2\tilde{E}\right)\tilde{\psi}^2 x^2 dx, \tag{41}$$

where all the potentials are assumed to be given by our regular solution. One considers the following two possibilities. If binding energy is positive then it is necessary to add a certain amount of energy to create a regular electric charge when coupling to the ψ field. In the opposite case binding energy gets released during the process, and the energy of the whole configuration is smaller than the sum of energies of the separate electric and scalar fields.

In our case, evaluating the binding energy on the approximate solution (31), (32), we obtain that it is negative-definite

$$\Delta\tilde{W} = -2^{-5/2}\sqrt{\pi}\left(\tilde{g}\tilde{E}\,\mathrm{e}^{3-\tilde{E}^2}\right)^2 + \mathcal{O}(\tilde{g}^4), \tag{42}$$

which means that the creation of the regular electric charge in the logarithmic model is energetically favorable.

Another way to study the stability of the solution is to write it as a solution of the Schrödinger equation for a fictitious particle

$$-\Delta\Psi + V_{\text{eff}}(x)\Psi = \varepsilon\Psi, \tag{43}$$

where $\Delta = d^2/dx^2$, $\varepsilon = \tilde{E}^2$ and the effective potential is derived as

$$V_{\text{eff}}(x) = -2\tilde{E}\tilde{\phi}(x) - \tilde{\phi}^2(x) - \ln[\tilde{\psi}^2(x)], \tag{44}$$

where the tilded potentials are given by our regular solution. According to (23), the asymptotic behavior of the solution implies that the effective potential $V_{\text{eff}} \propto x^2$ at large x, see also Fig. 5, and thus the "particle" is always localized in a finite region of x. With respect to the solution itself this means that it cannot spread or get destroyed when subjected to small perturbations.

5. LOGARITHMIC VERSUS QUARTIC POTENTIAL

One may wonder whether the singularity-free solution exists when the scalar sector of our model is controlled not by the logarithmic potential (6) but by the more orthodox one, such as the Higgs-type (quartic) potential:

$$V_H(\psi) = -\frac{\varkappa}{4}\psi^4 + \frac{m^2}{2}\psi^2. \tag{45}$$

Corresponding dimensionless equations with the ansatz (11) are

$$\tilde{\psi}'' + \frac{2}{x}\tilde{\psi}' = \left[\left(\tilde{E} + \tilde{\phi}\right)^2 - \lambda\tilde{\psi}^2 + 1\right]\tilde{\psi}, \tag{46}$$

$$\tilde{\phi}'' + \frac{2}{x}\tilde{\phi}' = 2\tilde{g}^2\left(\tilde{E} + \tilde{\phi}\right)\tilde{\psi}^2. \tag{47}$$

where we introduced following dimensionless quantities

$$x = mr; \quad \tilde{\psi} = \frac{\psi}{\psi(0)}; \quad \tilde{E} = \frac{E}{m}; \quad \tilde{\phi} = -\frac{g\phi}{m}; \quad \lambda = \frac{\psi(0)^2 \varkappa}{m^2}. \tag{48}$$

The boundary conditions are

$$\tilde{\psi}'(0) = 0; \quad \tilde{\phi}(0) = -1.04; \quad \tilde{\phi}'(0) = 0. \tag{49}$$

As in the logarithmic model, the regular solution exists only if the potential (45) opens down, i.e., when $\varkappa > 0$. The profiles of $\tilde{\psi}(x)$ and $\tilde{\phi}(x)$ in Fig. 7 are presented. For technical reasons in this case an eigenvalue is $\tilde{\psi}(0)$ not $\tilde{E}$. In Fig. 8 the profile of the electric field $\mathcal{E}$ is shown. In order to show that the electric field asymptotically has Coulomb behavior we present the profile $x^2\mathcal{E}(x)$ in Fig. 8. From these figures one can see that the qualitative behavior of the potential $\phi(x)$ and the electric field $\mathcal{E}(x)$ are the same as for the logarithmic potential.

The asymptotic behavior for the functions ψ and ϕ at large x is given by

$$\tilde{\psi}(x) \to \psi_\infty \frac{\mathrm{e}^{-x\sqrt{\tilde{E}+1}}}{x^2}, \tag{50}$$

$$\tilde{\phi}(x) \to -\frac{\tilde{q}}{x}, \tag{51}$$

where ψ_∞ is some constant. This still looks very similar to what we had earlier in the logarithmic case, however, if we study the stability of this solution then differences do arise. At first, if one computes the binding energy similarly to (41) then it turns out to be positive, therefore, the creation of the regular electric charge by coupling electrical field to the quartic scalar one is energetically unfavorable. Further, if one computes the fictitious-particle potential for this solution, cf. (44),

$$V_{\text{eff}}(x) = -2\tilde{E}\tilde{\phi}(x) - \tilde{\phi}^2(x) - \lambda\tilde{\psi}^2(x) + 1, \tag{52}$$

then one finds that it approaches a constant at large x, see also Fig. 9. Therefore, the "particle" is not necessarily localized in a finite region of x. With respect to the solution itself, this means that the latter can spread or become unstable against small perturbations.

6. CONCLUSION

The classical model of a spinless electrical charge is described as a self-consistent field configuration of the EM field interacting with the fluctuations of the nontrivial physical vacuum effectively represented by the logarithmic Bose-Einstein condensate. We have shown that a regular solution does exist - as opposite to the case of the EM field propagating in the absolutely empty space. In this regard we recall the state of affairs in quantum mechanics: the Dirac/Schrödinger equation without any external potential has the de Broglie wave solution, the Dirac/Schrödinger equation with the external electrostatic field yields the regular wave functions (the hydrogen atom being an example), but the Dirac/Schrödinger coupled to Maxwell equations do not lead to a regular stationary solution. The reason is that the Dirac/Schrödinger equation *ab initio* describes a point-like particle which might be a good approximation for long-wavelength measurements, but in higher-energy and shorter-length regimes this approximation eventually becomes too crude, since it neglects internal structure and non-zero spatial extent. Among other things, this leads to the densities of energy and charge becoming infinite at the particle's position. Here we have shown that by introducing an additional player on scene, the physical vacuum condensate, one can obtain a regular solution, thus endowing particles with internal structure and spatial extent. The solution turns out to be stable and energetically favorable. Using its features, some observational constraints for the parameters of the theory have been derived. We also specified the conditions under which our model can be (approximately) interpreted in terms of a scalar particle and those under which it cannot.

Further, we have established, both numerically and analytically, that the mass and spatial extent of a charged particle emerge due to the interaction of the EM field with the vacuum. It has been demonstrated that the average charge radius becomes non-zero, and the charge density acquires the Gaussian-like form. Looking at the form of the analytical solution from section 4.2, one can infer that it describes the object without border in a classical sense, therefore, its stability is supported not by surface tension but by nonlinear quantum effects in the bulk, similarly to the non-relativistic case [25]. Due to the non-singular behavior of the solution at the origin, the derivation of self-energy turned out to be entirely divergence-free.

The derived mass formula (38) suggested that for an electrically charged massive elementary particle there exists not only an antiparticle of the same mass (in the leading-order approximation with respect to the Planck constant, at least) but also a neutral particle of related mass. This might explain, at least qualitatively, why an electrically charged elementary particle is often accompanied by a neutral particle, and only one, of a similar kind – but not *vice versa*. Indeed, such "mass pairing" feature has been observed (modulo the influence of internal degrees of freedom such as spin, isospin, *etc.*) not only for the elementary particles, such as leptons and weak bosons, but also for stable composite ones such as nucleons (the quarks might not fit this scheme since they are confined inside hadrons). We gave some arguments for why the rest mass of a neutral partner can be sometimes so much smaller, yet non-vanishing, than the mass of the charged one (leptons), and sometimes they are of the same order of magnitude (weak bosons or nucleons).

Finally, we have compared the logarithmic vacuum model with the one based on the Higgs-type (inverted quartic) potential. It turns out that the corresponding regular solution is unstable and energetically unfavorable, in contrast with the logarithmic case.

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APPENDIX: DERIVATION OF APPROXIMATE SOLUTION

Here we provide more details regarding the derivation of the approximate analytical solution from section 4.2. Assuming (29) and (30), we will look for a solution of the system (16) and (17) in the series form

$$\tilde{\phi} = \tilde{\Phi}_0(x) + \xi \tilde{\Phi}_1(x) + \mathcal{O}(\xi^2), \quad \tilde{\psi} = \tilde{\Psi}_0(x) \left(1 + \xi \tilde{\Psi}_1(x) + \mathcal{O}(\xi^2)\right), \tag{A.1}$$

where $\xi = \tilde{g}^2$, and $\tilde{\Phi}_i(x)$ and $\tilde{\Psi}_k(x)$ are functions to be determined. Keeping terms of the order $\mathcal{O}(\xi)$ and below, the equations (16) and (17) can be reduced to the following set of four differential equations:

$$\tilde{\Phi}_0'' + \frac{2}{x}\tilde{\Phi}_0' = 0, \tag{A.2}$$

$$\tilde{\Psi}_0'' + \frac{2}{x}\tilde{\Psi}_0' + \left[\left(\tilde{E} + \tilde{\Phi}_0\right)^2 + \ln(\tilde{\Psi}_0^2)\right]\tilde{\Psi}_0 = 0, \tag{A.3}$$

$$\tilde{\Phi}_1'' + \frac{2}{x}\tilde{\Phi}_1' - 2\left(\tilde{E} + \tilde{\Phi}_0\right)\tilde{\Psi}_0^2 = 0, \tag{A.4}$$

$$\tilde{\Psi}_1'' + 2\left(\frac{1}{x} + \frac{\tilde{\Psi}_0'}{\tilde{\Psi}_0}\right)\tilde{\Psi}_1' + 2(\tilde{E} + \tilde{\Phi}_0)\tilde{\Phi}_1 + 2\tilde{\Psi}_1 = 0. \tag{A.5}$$

By solving them we obtain

$$\tilde{\phi} = \Lambda + \xi\left(c_2 - \frac{c_1}{x}\right) - \frac{\xi\sqrt{\pi}}{2}(\tilde{E} + \Lambda)\mathrm{e}^{3-(\tilde{E}+\Lambda)^2}\mathrm{erf}(x) + \mathcal{O}(\xi^2), \tag{A.6}$$

$$\tilde{\Psi}_0 = \mathrm{e}^{(3-(\tilde{E}+\Lambda)^2-x^2)/2}, \tag{A.7}$$

$$\begin{aligned}\tilde{\Psi}_1 = {} & \frac{1}{4}(\tilde{E} + \Lambda)^2\mathrm{e}^{(3-(\tilde{E}+\Lambda)^2-x^2)}\left[1 + \frac{\sqrt{\pi}}{2x}(2x^2+1)\mathrm{e}^{x^2}\mathrm{erf}(x)\right] - \\ & c_2(\tilde{E}+\Lambda) + c_3\left(x - \frac{1}{2x}\right) + c_4\left[\frac{\sqrt{\pi}}{2x}(2x^2-1)\mathrm{erfi}(x) - \mathrm{e}^{x^2}\right],\end{aligned} \tag{A.8}$$

where c_i and Λ are integration constants whose values must be fixed by means of the boundary conditions. Imposing (30), one obtains: $c_1 = c_3 = c_4 = 0$ and $c_2 = -\Lambda/\xi$. Then, after making the energy redefinition $\tilde{E} + \Lambda \to \tilde{E}$, one eventually arrives at the expressions (31) and (32).

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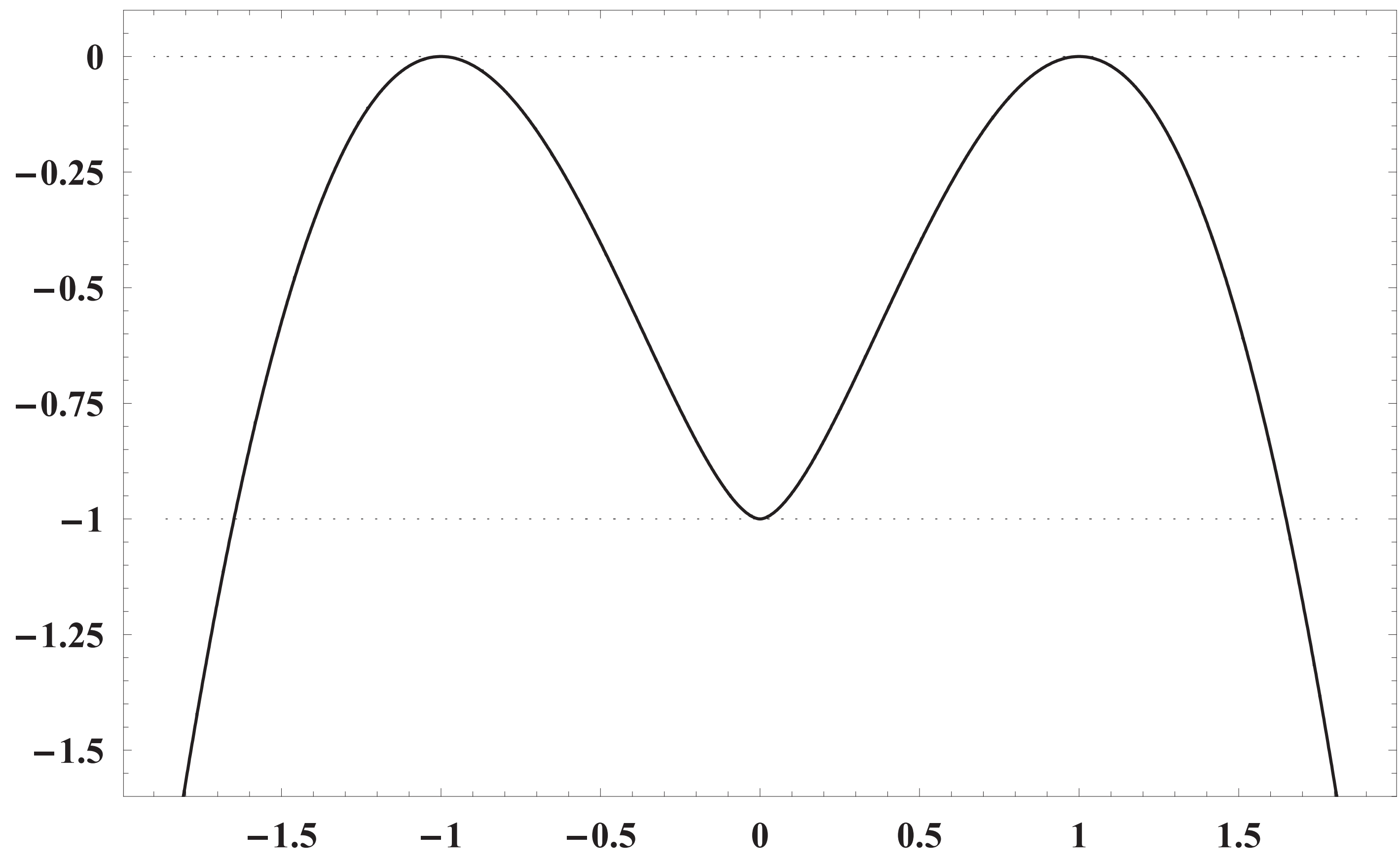

FIG. 1: The field-theoretical potential (2) (in units of $\breve{\beta}\breve{a}^3$) versus Re(Ψ) (in units of $\breve{a}^{-3/2}$). Vertical lines conditionally represent inequality (3). In an approximation when the symmetry-breaking energy scale is much less than the vacuum one ($\gtrsim$ 10 TeV) these walls can be assumed infinite.

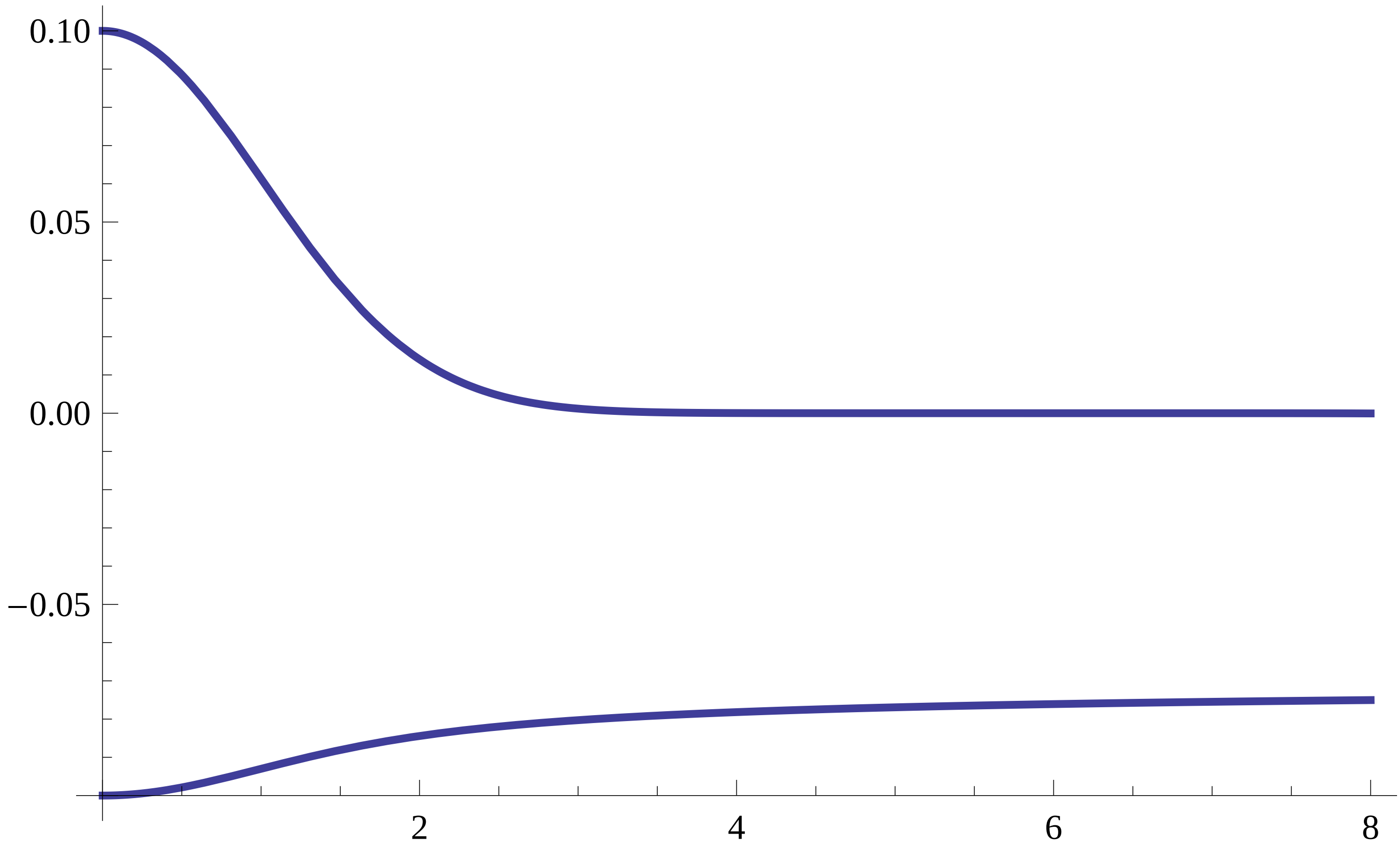

FIG. 2: The profiles of $\tilde{\psi}(x)$ and the electrostatic potential $\tilde{\phi}(x)$ (bottom curve), computed at the eigenvalue $\tilde{E} = 2.8436935588$.

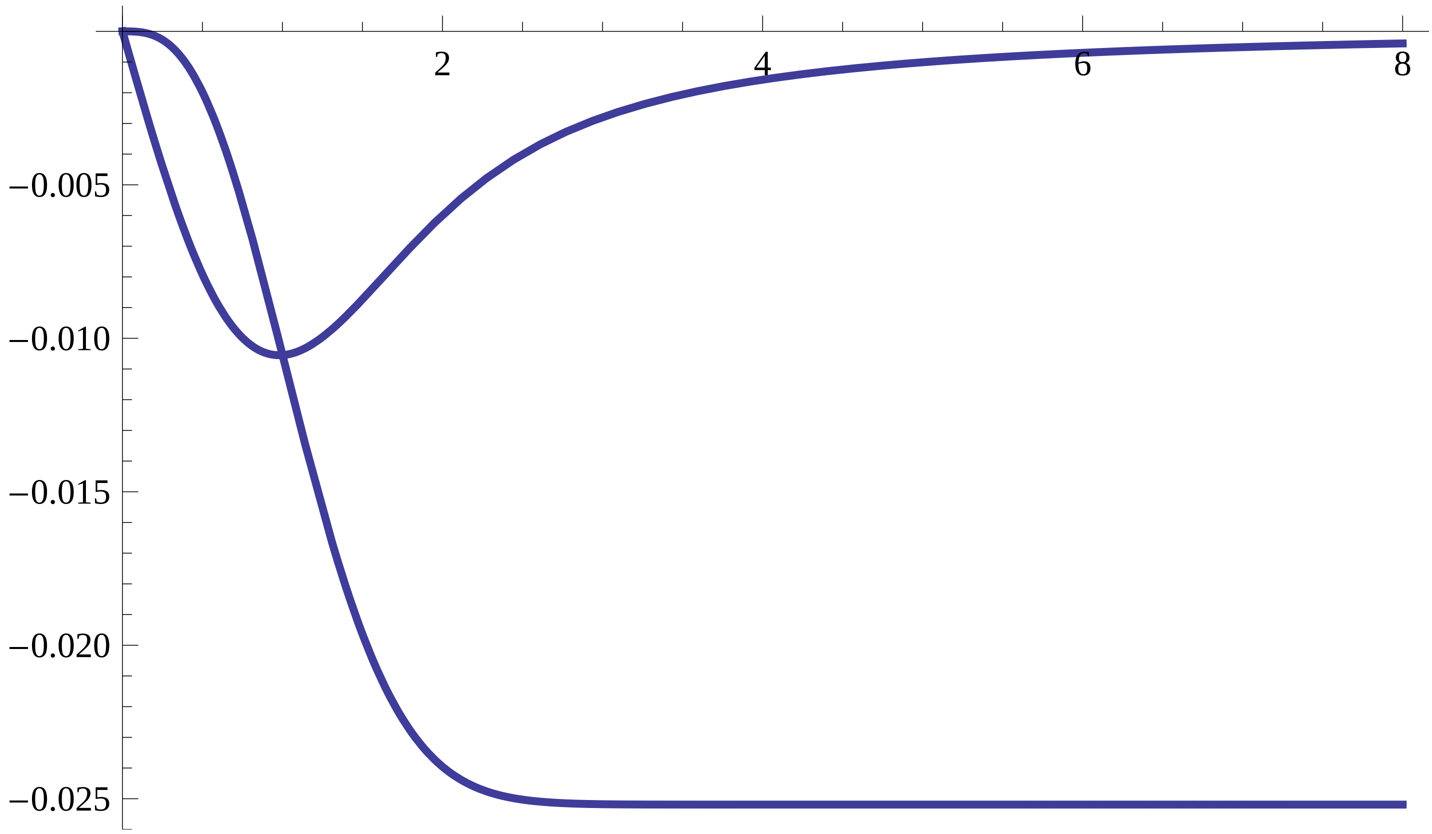

FIG. 3: The profiles of the electric field $\tilde{\mathcal{E}}(x)$ and $x^2\tilde{\mathcal{E}}(x)$ (bottom curve).

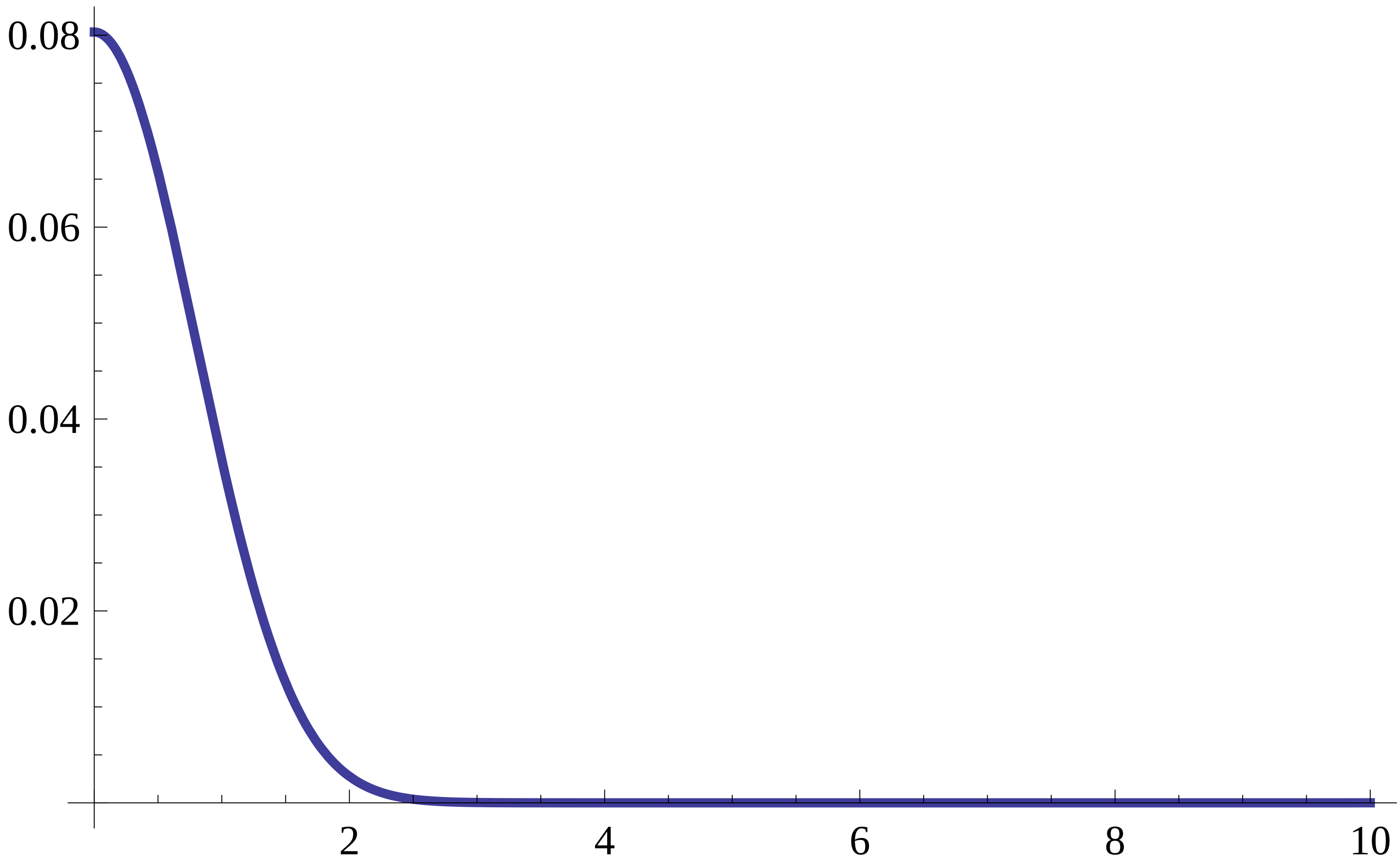

FIG. 4: The profile of dimensionless energy density $\beta a^3 \epsilon(x)$, same $\tilde{E}$ eigenvalue as in previous figures.

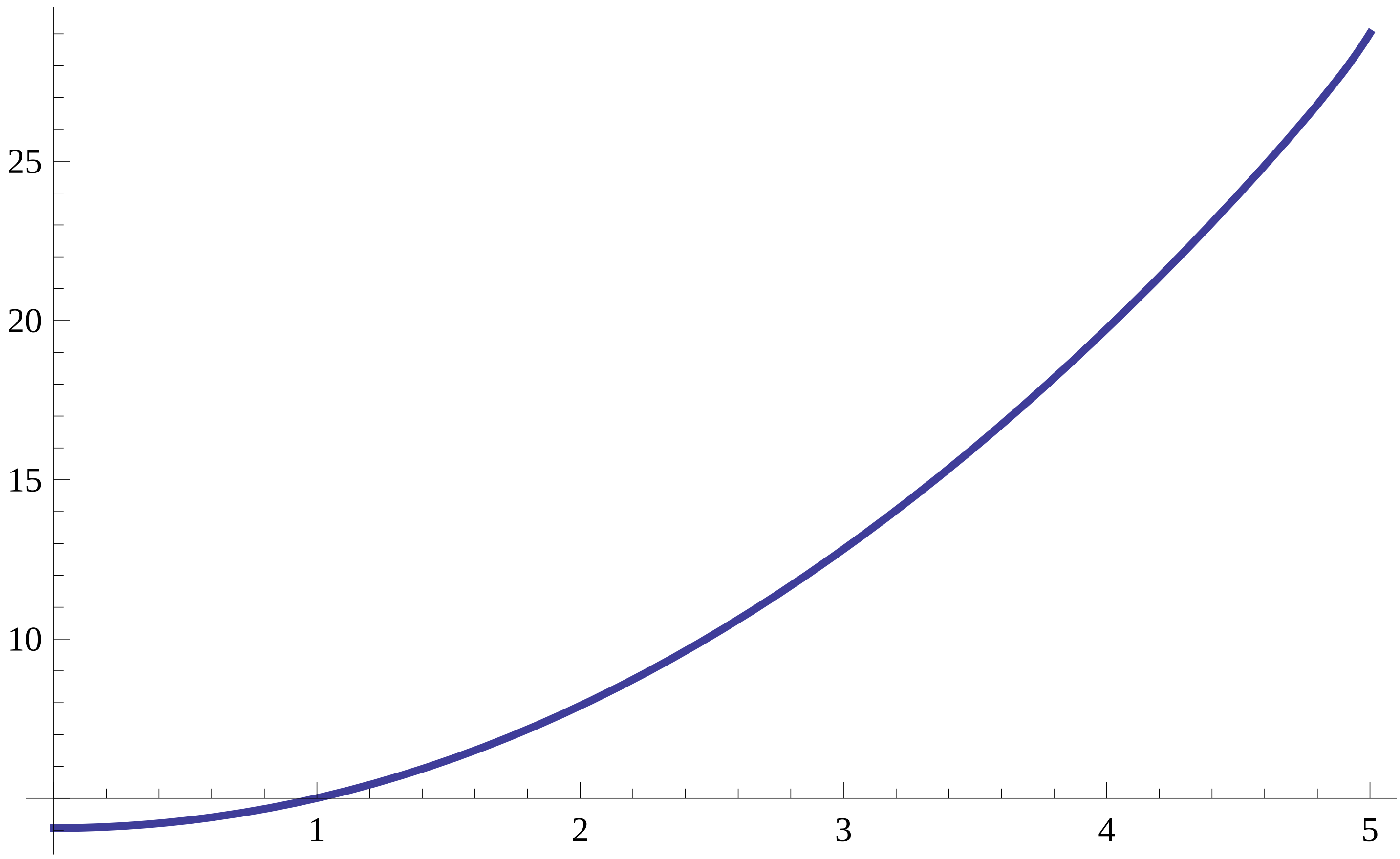

FIG. 5: Effective potential (44) versus x at $\tilde{g}$=1, same $\tilde{E}$ eigenvalue as in previous figures.

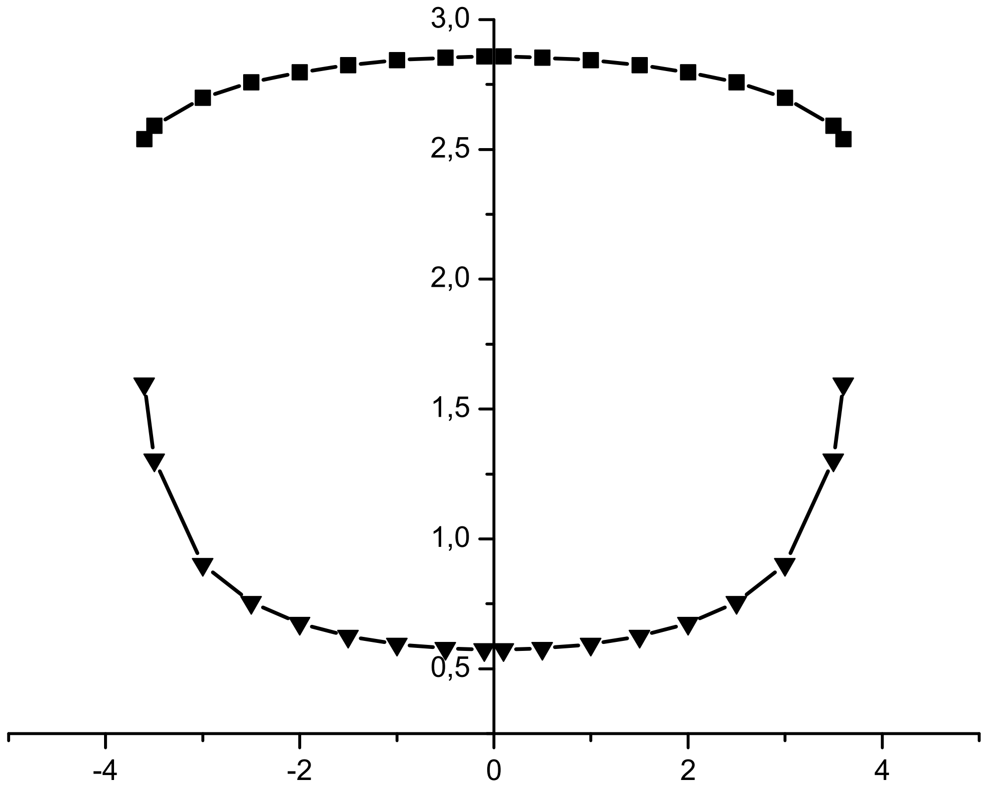

FIG. 6: The profiles of the dimensionless energy $\tilde{W}(\tilde{g})$ (▼-curve) and the eigenvalues $\tilde{E}(\tilde{g})$ (■ - curve).

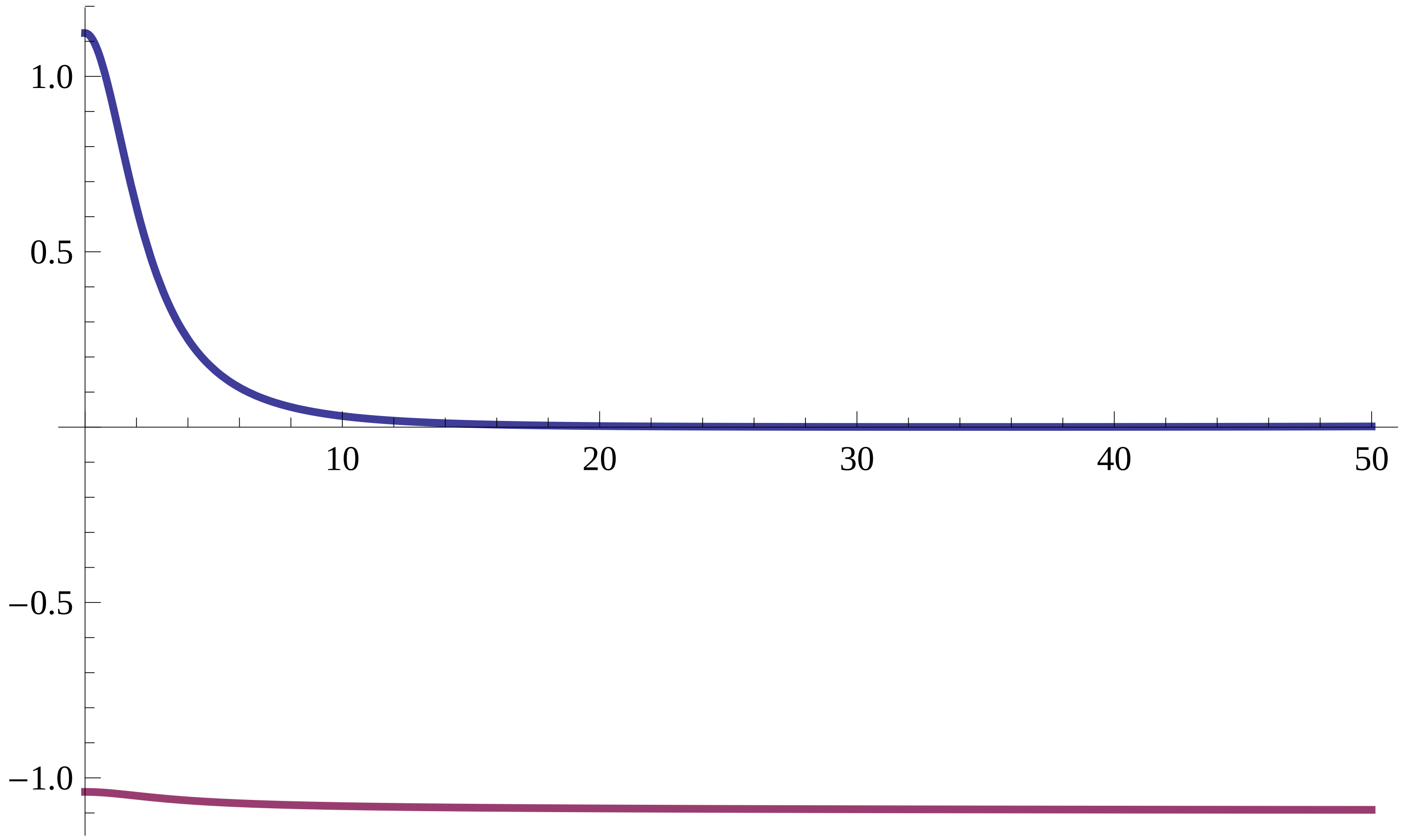

FIG. 7: The profiles of $\tilde{\psi}(x)$ (top) and $\tilde{\phi}(x)$ for the quartic model. The eigenvalue $\tilde{\psi}(0) = 1.12345$, $\tilde{g} = 0.1$, the parameters $\tilde{E} = 0.1, \lambda = 1.0$.

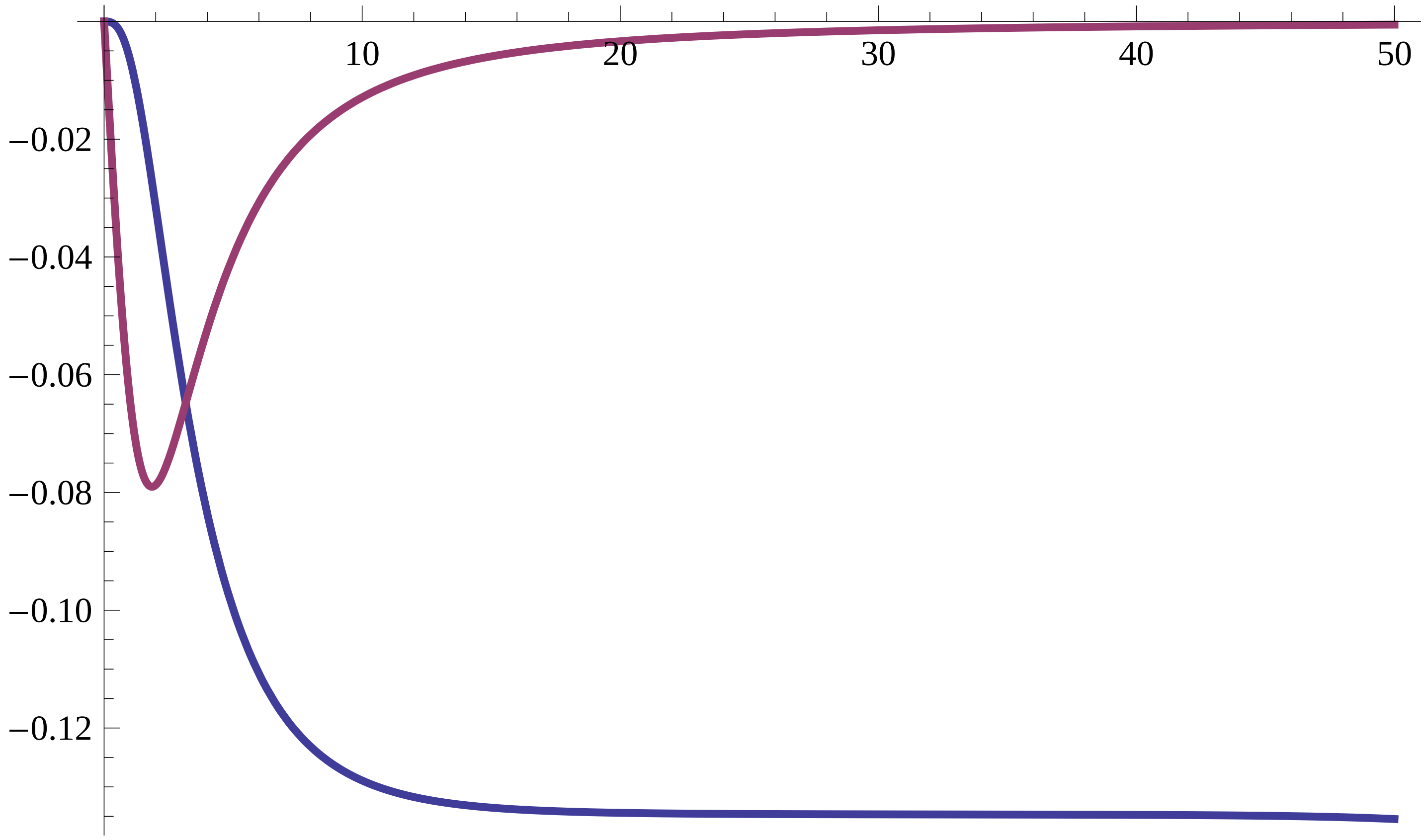

FIG. 8: Electric field versus x for the quartic model. The top curve is $10\,\tilde{\mathcal{E}}(x)$, the bottom one is $x^2\tilde{\mathcal{E}}(x)$.

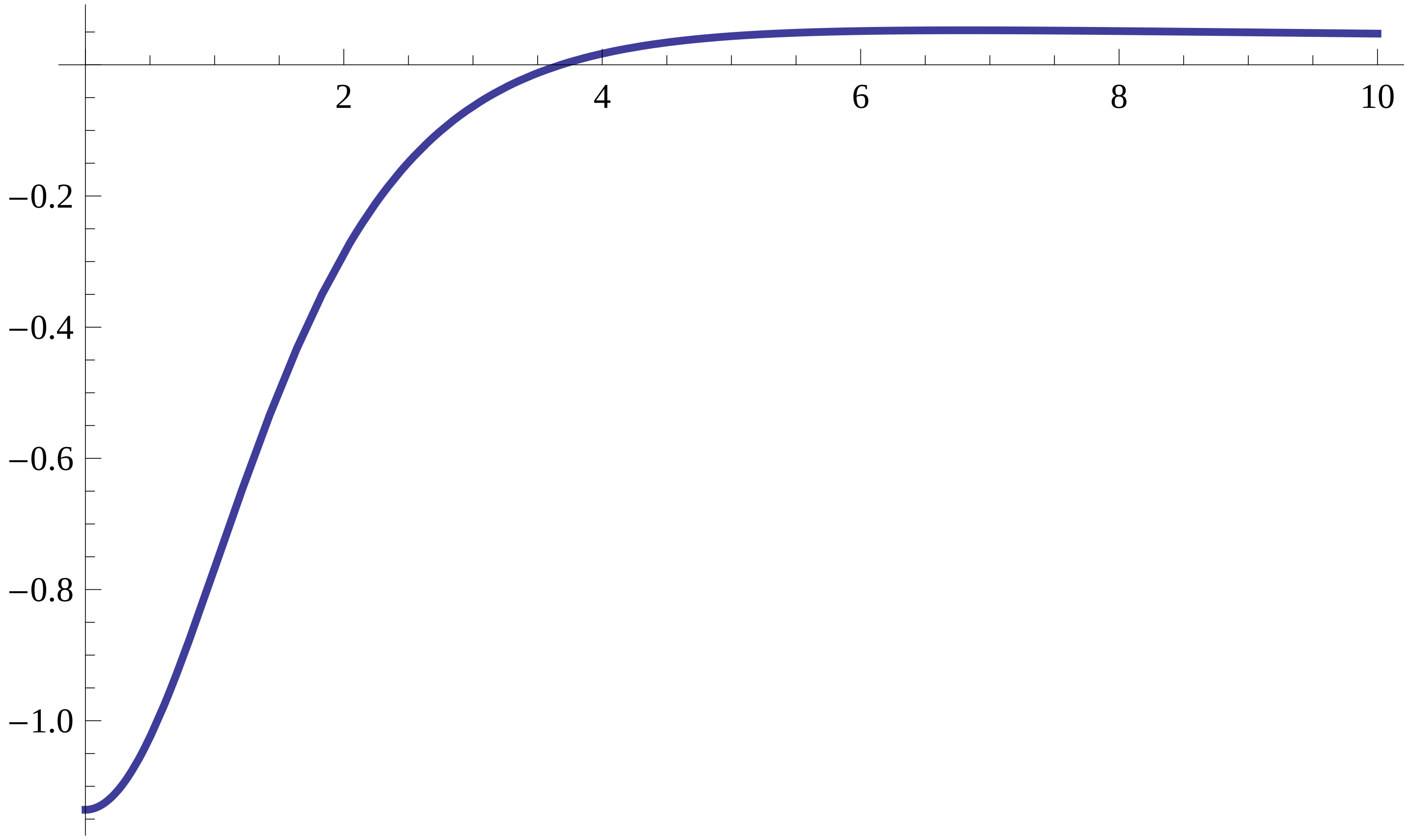

FIG. 9: Effective potential (52) versus x at $\tilde{g} = 1$ for the quartic model.