IOP Publishing Classical and Quantum Gravity
Class. Quantum Grav. 32 (2015) 165010 (15pp) doi:10.1088/0264-9381/32/16/165010

Dilatonic dyon black hole solutions

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Received 28 April 2015, revised 15 June 2015
Accepted for publication 24 June 2015
Published 27 July 2015

Abstract

Dilatonic black hole dyon solutions with an arbitrary dilatonic coupling constant $\lambda \neq 0$ and canonical sign $\varepsilon = +1$ for the scalar field kynetic term are considered. These solutions are defined up to solutions of two master equations for moduli functions. For $\lambda^2 \neq 1/2$ the solutions are extended to $\varepsilon = \pm 1$, where $\varepsilon = -1$ corresponds to a ghost (phantom) scalar field. Some physical parameters of the solutions: gravitational mass, scalar charge, Hawking temperature, black hole area entropy and parametrized post-Newtonian (PPN) parameters β and γ are obtained. It is shown that PPN parameters do not depend on scalar field coupling λ and ε. Two groups of bounds on gravitational mass and scalar charge (for a fixed and arbitrary extremality parameter $\mu > 0$) are found by using a certain conjecture on parameters of solutions when $1 + 2\lambda^2\varepsilon > 0$. These bounds are verified numerically for certain examples. Using the product we are led to the well-known lower bound on mass which was obtained earlier by Gibbons, Kastor, London, Townsend and Traschen by using spinor techniques.

Keywords: black holes, dyon, mass bounds

(Some figures may appear in colour only in the online journal)

1. Introduction

At present there exists an interest in spherically-symmetric solutions, e.g. black hole and black brane ones, related to Lie algebras and Toda chains, see [1–23] and references therein.

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These solutions appear in gravitational models with scalar fields and antisymmetric forms. Meanwhile special subclasses of low-dimensional (e.g. 4-dimensional) solutions were not considered in detail.

Here we consider a subclass of 4-dimensional dilatonic black hole solutions with electric and magnetic charges. We extend the dilatonic black hole dyon solution from [18] to a more general case when the dilatonic scalar field may be a ghost (or phantom) one. The ghost field appears in the action with a kinetic term of the 'wrong sign.' This implies the violation of the null energy condition $p + \rho \geqslant 0$. At the quantum level, such fields could form a 'ghost condensate,' which would be responsible for modified gravity laws in the infra-red limit [24]. Present observational data do not exclude this possiblity, and moreover, under certain conditions the phantom scenario has a preference [25].

The main goal of our paper is the search for relations on physical parameters of dyonic black holes, e.g. bounds on gravitational mass M and scalar charge Q_φ. This problem is solved here up to a conjecture, which states one to one (smooth) correspondence between the pair (Q_1^2, Q_2^2), where Q_1 is electric charge and Q_2 is magnetic charge, and the pair (P_1, P_2), where $P_1 > 0$ and $P_2 > 0$ are parameters of the solutions. This conjecture is believed to be valid for all $\lambda \neq 0$ in the case of ordinary scalar field and for $0 < \lambda^2 < 1/2$ for the case of a phantom scalar field. Here we verify the bounds by using numerical calculations.

2. Black hole dyon solutions

2.1. Dyonic solutions

We consider a model governed by the action

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\left\{R[g] - \varepsilon g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\exp(2\lambda\varphi)F_{\mu\nu}F^{\mu\nu}\right\},$$

where $g = g_{\mu\nu}(x)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ is metric, φ is the scalar field, $F = \mathrm{d}A = \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is the Maxwell 2-form, $A = A_\mu \mathrm{d}x^\mu$, $\varepsilon = \pm 1$, G is the gravitational constant, $\lambda \neq 0$ is a coupling constant and $|g| = |\det(g_{\mu\nu})|$. Here we put $\lambda^2 \neq 1/2$ for $\varepsilon = -1$.

Let us consider a family of dyonic black hole solutions to the field equations corresponding to the action (2.1). These solutions are defined on the manifold

$$M = (2\mu, +\infty) \times S^2 \times \mathbb{R}, \tag{2.1}$$

and have the following form

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = (H_1H_2)^h\left\{-(H_1H_2)^{-2h}\left(1 - \frac{2\mu}{R}\right)\mathrm{d}t^2 + \frac{\mathrm{d}R^2}{1 - \frac{2\mu}{R}} + R^2\mathrm{d}\Omega_2^2\right\}, \tag{2.2}$$

$$\exp(\varphi) = \left(\frac{H_1}{H_2}\right)^{h\lambda\varepsilon}, \tag{2.3}$$

$$F = \frac{Q_1}{R^2}H_1^{-2}H_2^{-a}\mathrm{d}t \wedge \mathrm{d}R + Q_2\tau. \tag{2.4}$$

Here Q_1 and Q_2 are electric and magnetic charges, respectively, $\mu > 0$ is the extremality parameter, $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the canonical metric on the unit sphere S^2 ($0 < \theta < \pi$, $0 < \phi < 2\pi$), $\tau = \sin\theta d\theta \wedge d\phi$ is the standard volume form on S^2,

$$h = \frac{2}{1 + 2\lambda^2\varepsilon}, \tag{2.5}$$

and

$$a = 2\frac{1 - 2\lambda^2\varepsilon}{1 + 2\lambda^2\varepsilon}. \tag{2.6}$$

Functions $H_s > 0$ obey the equations

$$R^2\frac{d}{dR}\left(R^2\frac{\left(1 - \frac{2\mu}{R}\right)}{H_s}\frac{dH_s}{dR}\right) = -h^{-1}Q_s^2 \prod_{l=1,2} H_l^{-A_{sl}}, \tag{2.7}$$

with the following boundary conditions imposed

$$H_s \to H_{s0} > 0, \tag{2.8}$$

for $R \to 2\mu$, and

$$H_s \to 1, \tag{2.9}$$

for $R \to +\infty$, $s = 1, 2$.

In equation (2.10) we denote

$$(A_{ss'}) = \begin{pmatrix} 2 & a \\ a & 2 \end{pmatrix}. \tag{2.10}$$

For the case $\varepsilon = 1$ these solutions were presented earlier in [18]. They may be obtained by using general formulae for non-extremal (intersecting) black brane solutions from [14–17].

The first boundary condition (2.8) guarantees (up to a possible additional demand on analicity of $H_s(R)$ in the vicinity of $R = 2\mu$) the existence of a (regular) horizon at $R = 2\mu$ for the metric (2.2). The second condition (2.9) ensures an asymptotical (for $R \to +\infty$) flatness of the metric.

Equations (2.7) may be rewritten in the following form

$$\frac{d}{dz}\left[(1 - z)\frac{dy_s}{dz}\right] = -bq_s^2 \exp\left(-2y_s - ay_{\bar{s}}\right), \tag{2.11}$$

$s = 1, 2$. Here and in what follows we use the following notations: $y_s = \ln H_s$, $z = 2\mu/R$, $q_s = Q_s/(2\mu)$, $b = h^{-1}$ and $\bar{s} = 2, 1$ for $s = 1, 2$, respectively. We are seeking solutions to equations (2.11) for $z \in (0, 1)$ obeying

$$y_s(0) = 0, \tag{2.12}$$

$$y_s(1) = y_{s0}, \tag{2.13}$$

where $y_{s0} = \ln H_{s0}$ are finite (real) numbers, $s = 1, 2$. Here $z = 0$ (or, more precisely $z = +0$) corresponds to infinity ($R = +\infty$), while $z = 1$ (or, more rigorously, $z = 1 - 0$) corresponds to the horizon ($R = 2\mu$).

Equations (2.11) with the finite conditions on the horizon (2.13) imposed imply the following integral of motion:

$$(1 - z)\left[\left(\frac{dy_1}{dz}\right)^2 + \left(\frac{dy_2}{dz}\right)^2 + a\frac{dy_1}{dz}\frac{dy_2}{dz}\right] + \frac{dy_1}{dz} + \frac{dy_2}{dz}$$
$$- bq_1^2 \exp\left(-2y_1 - ay_2\right) - bq_2^2 \exp\left(-2y_2 - ay_1\right) = 0. \tag{2.14}$$

Equations (2.11) and (2.13) appear for special solutions to Toda-type equations [15–17]

$$\frac{d^2 z_s}{du^2} = bQ_s^2 \exp\left(2z_s + az_{\bar{s}}\right), \tag{2.15}$$

for functions $z_s(u) = -y_s - \mu bu$, $s = 1, 2$, depending on the harmonic radial variable u: $\exp(-2\mu u) = 1 - z$, with the following asymptotical behaviour for $u \to +\infty$ (on the horizon) imposed:

$$z_s(u) = -\mu bu + z_{s0} + o(1), \tag{2.16}$$

where z_{s0} are constants, $s = 1, 2$. The energy integral of motion for equation (2.15), which is compatible with the asymptotic conditions (2.16),

$$E = \frac{h}{2}\left[\left(\frac{dz_1}{du}\right)^2 + \left(\frac{dz_2}{du}\right)^2 + a\frac{dz_1}{du}\frac{dz_2}{du}\right]$$
$$- \sum_{s=1,2} \frac{1}{2} Q_s^2 \exp\left(2z_s + az_{\bar{s}}\right) = \mu^2, \tag{2.17}$$

leads us to relation (2.14).

Remark 1. Here we exclude the case $\lambda^2 = 1/2$ for $\varepsilon = -1$ from our consideration since we deal with the finite value of the parameter h. But nevertheless, one can also obtain a sensible solution with a horizon for this peculiar case. This may be achieved by using another choice of moduli functions: $\bar{H}_s = H_s^h$ instead of H_s. The (implicit) solutions given by (2.2), (2.3), and (2.4) and the master equations (2.7), rewritten in terms of new moduli functions $\bar{H}_s$, have a sensible limit for $\lambda^2 = 1/2$ and $\varepsilon = -1$. This special case may be the subject of a separate publication.

3. Some integrable cases

At present it seems impossible to find explicit solutions to the equations (2.7)–(2.9) analytically. One may try to seek the solutions in the form

$$H_s = 1 + \sum_{k=1}^{\infty} P_s^{(k)} \left(\frac{1}{R}\right)^k, \tag{3.1}$$

where $P_s^{(k)}$ are constants, $k = 1, 2, \ldots$ and $s = 1, 2$.

Remark 2. The $1/R$ expansion is widely used in gravitational physics sometimes without any indication whether the series like equation (3.1) is i) convergent or ii) an asymptotical one. Here the first possibility i) follows from analytical behaviour of functions H_s with respect to $z = 2\mu/R$ in the vicinity of the point $z = 0$, which is based on equations (2.14) and the non-degeneracy condition for the function $f(z) = 1 - z$ at $z = 0$, i.e. $f(0) = 1 \neq 0$. In what follows any of two assumptions: i) or ii) (or even a more modest one: $H_s = 1 + P_s/R + o(1/R)$, for $R \to +\infty$) is enough for our analysis.

Meanwhile, there exist at least two integrable configurations related to Lie algebras $A_1 + A_1$ and A_2.

3.1. $(A_1 + A_1)$-case

Let

$$\lambda^2 = \frac{1}{2}, \qquad \varepsilon = 1. \tag{3.2}$$

This value of dilatonic coupling corresponds to a string induced model. We get $h = 1$, $a = 0$ and hence equation (2.10) is the Cartan matrix for the Lie algebra $A_1 + A_1$ ($A_1 = sl(2)$). In this case

$$H_s = 1 + \frac{P_s}{R}, \tag{3.3}$$

where

$$P_s(P_s + 2\mu) = Q_s^2, \tag{3.4}$$

$s = 1, 2$. For positive roots of equation (3.4)

$$P_s = P_{s+} = -\mu + \sqrt{\mu^2 + Q_s^2}, \tag{3.5}$$

we are led to a well-defined $R > 2\mu$ solution with an asymptotically flat metric and horizon at $R = 2\mu$. We note that an $(A_1 + A_1)$-dyon solution was considered earlier in [6, 8]; see also [12, 19] for certain extensions.

3.2. A_2-case

Let

$$\lambda^2 = 3/2, \qquad \varepsilon = 1. \tag{3.6}$$

This value of the dilatonic coupling constant appears after reduction to four dimensions of the 5-dimensional Kaluza–Klein model. We get $h = 1/2$, $a = -1$ and equation (2.10) is the Cartan matrix for the Lie algebra $A_2 = sl(3)$. In this case we obtain [15]

$$H_s = 1 + \frac{P_s}{R} + \frac{P_s^{(2)}}{R^2}, \tag{3.7}$$

where

$$2Q_s^2 = \frac{P_s(P_s + 2\mu)(P_s + 4\mu)}{P_1 + P_2 + 4\mu}, \tag{3.8}$$

$$P_s^{(2)} = \frac{P_s(P_s + 2\mu)P_{\bar{s}}}{2(P_1 + P_2 + 4\mu)}, \tag{3.9}$$

$s = 1, 2$ ($\bar{s} = 2, 1$). The Kaluza–Klein uplift to $D = 5$ gives us the well-known Gibbons–Wilthire solution [4], which is in agreement with the general spherically-symmetric dyon solution (related to A_2 Toda chain) from [3].

3.3. Special solution with equal charges

There exists also a special solution

$$H_s = \left(1 + \frac{P}{R}\right)^b, \tag{3.10}$$

with equal charges $Q_s = Q$, $s = 1, 2$, satisfying

$$Q^2 = P(P + 2\mu). \tag{3.11}$$

We remind the reader that $b = h^{-1}$. For the positive root of equation (3.11)

$$P = P_+ = -\mu + \sqrt{\mu^2 + Q^2}, \tag{3.12}$$

we get for $R > 2\mu$ a well-defined solution with an asymptotically flat metric and horizon at $R = 2\mu$.

This solution is a special case of more general 'block orthogonal' black brane solutions [26, 27]. Here the power in equation (3.10) appears due to the relation

$$b = 2\sum_{l=1,2} A^{sl}, \tag{3.13}$$

$s = 1, 2$, where $(A^{sl}) = (A_{sl})^{-1}$. This power is an integer for the $A_1 + A_1$ and A_2 cases.

It should be noted that this special solution is valid for both signs $\varepsilon = \pm 1$ and has a well-defined limit for $\lambda^2 = 1/2$, $\varepsilon = -1$ in agreement with remark 1 (here $\bar{H}_s = 1 + P/R$).

3.4. The limiting A_1-case

In what follows we will use two limiting solutions: the electric one with $Q_1 = Q \neq 0$ and $Q_2 = 0$,

$$H_1 = 1 + \frac{P}{R}, \qquad H_2 = 1, \tag{3.14}$$

and the magnetic one with $Q_1 = 0$ and $Q_2 = Q \neq 0$,

$$H_1 = 1, \qquad H_2 = 1 + \frac{P}{R}. \tag{3.15}$$

In both cases $P = -\mu + \sqrt{\mu^2 + bQ^2}$. These solutions correspond to the Lie algebra A_1. In various notations the solution (3.14) appeared earlier in [1] and [5, 6], and was extended to the multidimensional case in [5, 6, 9, 10][1]. A special case with $\lambda^2 = 1/2$, $\varepsilon = 1$ was considered earlier in [2, 7].

4. Physical parameters

Here we consider certain physical parameters corresponding to the solutions under consideration.

[1] The results of [6] seem to be correct ones up to a typo in the first formula (2.1) for the action in [6] which should be eliminated: the kinetic term for the scalar field should be multiplied by an extra factor of 1/2.

4.1. Gravitational mass and scalar charge

For (ADM) gravitational mass we get from equation (2.2)

$$GM = \mu + \frac{h}{2}(P_1 + P_2), \tag{4.1}$$

where parameters $P_s = P_s^{(1)}$ appear in the relation (3.1) and G is the gravitational constant.

The scalar charge just follows from equation (2.3)

$$Q_\varphi = \lambda h \varepsilon (P_1 - P_2). \tag{4.2}$$

For the symmetric case $Q_1^2 = Q_2^2 = Q^2 = P(P + 2\mu)$ with $P > 0$ we get $P_1 = P_2 = bP$ and hence

$$GM = \mu + P = \sqrt{\mu^2 + Q^2}, \qquad Q_\varphi = 0. \tag{4.3}$$

In this case the gravitational mass and the scalar charge do not depend upon λ and ε. The mass M monotonically increases from μ (for $Q^2 = +0$) to $+\infty$ (for $Q^2 = +\infty$).

For fixed charges Q_s and the extremality parameter μ the mass M and scalar charge Q_φ are not independent but obey a certain constraint. Indeed, for fixed parameters $P_s = P_s^{(1)}$ in decomposition (3.1) we get

$$y_s = \ln H_s = \frac{P_s}{2\mu} z + O\left(z^2\right), \tag{4.4}$$

for $z \to +0$, which after substitution into equation (2.14) gives us (for $z = 0$) the following identity

$$P_1^2 + P_2^2 + aP_1P_2 + 2\mu(P_1 + P_2) = b\left(Q_1^2 + Q_2^2\right). \tag{4.5}$$

By using relations (4.1) and (4.2) this identity may be rewritten in the following form

$$2(GM)^2 + \varepsilon Q_\varphi^2 = Q_1^2 + Q_2^2 + 2\mu^2. \tag{4.6}$$

It is remarkable that this formula does not contain λ. We note that in the extremal case $\mu = +0$ this relation for $\varepsilon = 1$ was obtained earlier in [11]. In the derivation of equation (4.6) the following identities were used

$$a + 2 = 2h, \qquad 2 - a = 4\varepsilon\lambda^2 h. \tag{4.7}$$

Remark 3. The paper [11] is an important one due to the following non-trivial result which was obtained numerically: for $\varepsilon = 1$ the global extension of the metric (2.2) has two horizons only if $\lambda^2 = p(p + 1)/4$, $p = 1, 2, \ldots$. Recently, this rule was explained in [23] in terms of analyticity of the dilaton at the $AdS^2 \times S^2$ event horizon.

4.2. The Hawking temperature and entropy

The Hawking temperature corresponding to the solution is found to be

$$T_H = \frac{1}{8\pi\mu}\left(H_{10}H_{20}\right)^{-h}, \tag{4.8}$$

where H_{s0} are defined in equation (2.8). Here and in what follows we put $c = \hbar = \kappa = 1$.

For the symmetric case $Q_1^2 = Q_2^2 = Q^2 = P(P + 2\mu)$ with $P > 0$ we get

$$T_H = \frac{1}{8\pi\mu}\left(1 + \frac{P}{2\mu}\right)^{-2}. \tag{4.9}$$

We see that in this case the Hawking temperature T_H does not depend upon the choice of λ and ε. It monotonically decreases from $1/(8\pi\mu)$ (for $Q^2 = +0$) to 0 (for $Q^2 = +\infty$). (Here μ is fixed.)

The Bekenstein–Hawking (area) entropy $S = A/(4G)$, corresponding to the horizon at $R = 2\mu$, where A is the horizon area, reads

$$S_{\rm BH} = \frac{4\pi\mu^2}{G}\left(H_{10}H_{20}\right)^h. \tag{4.10}$$

It follows from equation (4.8) and (4.10) that the product

$$T_H S_{\rm BH} = \frac{\mu}{2G} \tag{4.11}$$

does not depend upon λ, ε and charges Q_s. This product does not use an explicit form of the moduli functions $H_s(R)$.

4.3. PPN parameters

Now we introduce a new radial variable ρ by the relation $R = \rho(1 + (\mu/2\rho))^2$ $(\rho > \mu/2)$, which gives us the 3-dimensionally conformally-flat form of the metric (2.2)

$$g^{(4)} = U\left\{-U_1\frac{(1-(\mu/2\rho))^2}{(1+(\mu/2\rho))^2}{\rm d}t \otimes {\rm d}t + \left(1 + \frac{\mu}{2\rho}\right)^4 \delta_{ij}{\rm d}x^i \otimes {\rm d}x^j\right\}, \tag{4.12}$$

where $\rho^2 = |x|^2 = \delta_{ij}x^i x^j$ $(i, j = 1, 2, 3)$ and

$$U = \prod_{s=1,2} H_s^h, \qquad U_1 = \prod_{s=1,2} H_s^{-2h}. \tag{4.13}$$

The parametrized post-Newtonian (PPN) parameters β and γ are defined by the following standard relations

$$g_{00}^{(4)} = -\left(1 - 2V + 2\beta V^2\right) + O\left(V^3\right), \tag{4.14}$$

$$g_{ij}^{(4)} = \delta_{ij}(1 + 2\gamma V) + O\left(V^2\right), \tag{4.15}$$

$i, j = 1, 2, 3$, where, $V = GM/\rho$ is Newton's potential, G is the gravitational constant and M is the gravitational mass (in our case given by equation (4.1)).

The calculations of PPN (or Edington) parameters for the metric (4.12) give us the same result as in [18]:

$$\beta = 1 + \frac{1}{4(GM)^2}\left(Q_1^2 + Q_2^2\right), \qquad \gamma = 1. \tag{4.16}$$

These parameters do not depend upon λ and ε. They may be calculated just without knowledge of explicit relations for functions $H_s(R)$.

It should be noted that (at least formally) these parameters obey the observational restrictions for the solar system [29] when the ratios $Q_s/(2GM)$ are small enough.

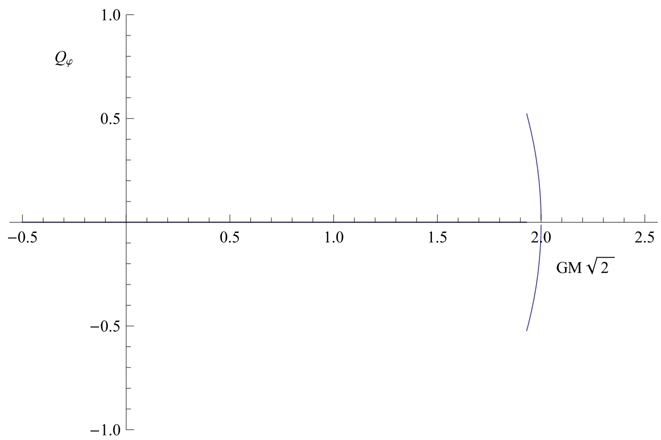

Figure 1. Graphical illustration of bounds on M and Q_φ for $\varepsilon = 1$, $\lambda = 1/\sqrt{2}$, $\mu = 1$ and $Q_1^2 + Q_2^2 = 2$.

5. Bounds on mass and scalar charge and numerical calculations

Here we start with the following hypothesis which is supported by numerical calculations.

Conjecture. For any $h > 0$, $\varepsilon = \pm 1$, $Q_1 \neq 0$, $Q_2 \neq 0$ and $\mu > 0$: A) the moduli functions $H_s(R)$, which obey (2.7), (2.8) and (2.9), are uniquely defined and hence the parameters P_1, P_2, the gravitational mass M and the scalar charge Q_φ are uniquely defined too; B) the parameters P_1, P_2 are positive and the functions $P_1 = P_1(Q_1^2, Q_2^2)$, $P_2 = P_2(Q_1^2, Q_2^2)$ define a diffeomorphism of $\mathbb{R}_+^2$ ($\mathbb{R}_+ = \{x | x > 0\}$); C) in the limiting case we have (i) for $Q_2^2 \to +0$: $P_1 \to -\mu + \sqrt{\mu^2 + bQ_1^2}$, $P_2 \to +0$ and (ii) for $Q_1^2 \to +0$: $P_1 \to +0$, $P_2 \to -\mu + \sqrt{\mu^2 + bQ_2^2}$ $(b = h^{-1})$.

The conjecture could be readily verified for the case $\varepsilon = 1$, $\lambda^2 = 1/2$. Another integrable case $\varepsilon = 1$, $\lambda^2 = 3/2$ is more involved and it needs some effort to verify this conjecture.

It seems that the point B) is at the same time the most crucial and most difficult to prove. This is the main part of the conjecture.

For $h > 0$ we are led to the following bounds on the gravitational mass M and scalar charge Q_φ ($Q_1 \neq 0$, $Q_2 \neq 0$)

$$\mu + \frac{h}{2}\left(-\mu + \sqrt{b\left(Q_1^2 + Q_2^2\right) + \mu^2} \right) < GM \leqslant \sqrt{\frac{1}{2}\left(Q_1^2 + Q_2^2\right) + \mu^2}, \tag{5.1}$$

for $\varepsilon = +1$ ($\lambda \neq 0$, $0 < h < 2$),

$$\sqrt{\frac{1}{2}\left(Q_1^2 + Q_2^2\right) + \mu^2} \leqslant GM < \mu + \frac{h}{2}\left(-\mu + \sqrt{b\left(Q_1^2 + Q_2^2\right) + \mu^2} \right), \tag{5.2}$$

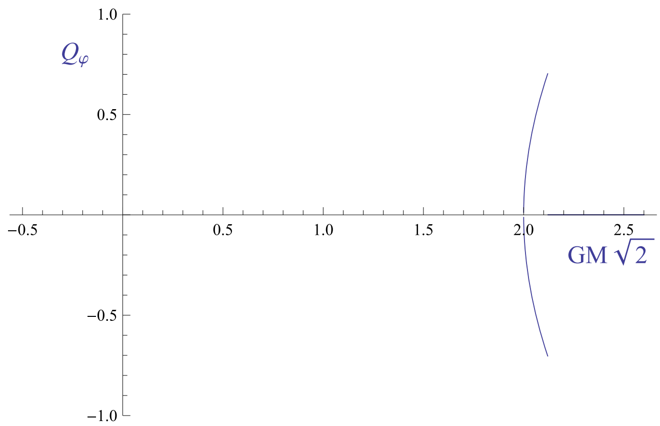

Figure 2. Graphical illustration of bounds on M and Q_φ for $\varepsilon = -1$, $\lambda = \sqrt{0.499}$, $\mu = 1$ and $Q_1^2 + Q_2^2 = 2$.

for $\varepsilon = -1$ $(0 < \lambda^2 < \frac{1}{2}, h > 2)$ and

$$|Q_\varphi| < |\lambda| h \left(-\mu + \sqrt{b\left(Q_1^2 + Q_2^2\right) + \mu^2} \right), \tag{5.3}$$

which are valid for all $\lambda \neq 0$.

We illustrate the bounds on M and Q_φ graphically by two figures, which represent a set of physical parameters GM and Q_φ for $Q_1^2 + Q_2^2 = Q^2 = 2$ and $\mu = 1$. Figure 1 corresponds to the case $\varepsilon = +1$ and $\lambda = \sqrt{\frac{1}{2}}$, while figure 2 describes the limiting case $\varepsilon = -1$ and $\lambda = \sqrt{0.499}$. The middle points of these two arcs correspond to symmetric solutions with $Q_1^2 = Q_2^2 = 1$, while the boundary points of the arcs correspond either to $Q_1^2 = +0$, $Q_2^2 = 2$, or to $Q_1^2 = 2$, $Q_2^2 = +0$.

Proof of the bounds. Let us prove the relations (5.1)–(5.3) using the conjecture. The right inequality (or equality) in equation (5.1) just follows from the relation (4.6), while the left inequality (or equality) in equation (5.2) follows from equation (4.6), and $M > 0$ which is valid due to relation (4.1), $h > 0$ and inequalities $P_1 > 0$, $P_2 > 0$ (due to the conjecture.). Now let us verify the left inequality in equation (5.1). We fix the charges by the relation $Q_1^2 + Q_2^2 = Q^2$, $Q > 0$, and put $Q_1^2 = \frac{1}{2}Q^2(1 + x)$, $Q_2^2 = \frac{1}{2}Q^2(1 - x)$, where $-1 < x < 1$. Due to the equation (4.6) and $M > 0$ we can use the following parametrization

$$\sqrt{2}\, GM = R \cos \psi, \qquad Q_\varphi = R \sin \psi, \qquad R = \sqrt{Q^2 + 2\mu^2}, \tag{5.4}$$

where $|\psi| < \pi/2$. Owing to conjecture and relations (4.1), (4.2) we get that $\psi = \psi(x)$ is a smooth function which obeys

$$\psi(0) = 0, \qquad \psi(\pm 1 \mp 0) = \pm \psi_0. \tag{5.5}$$

Here $R \cos \psi_0 = \sqrt{2}\,(\mu + \frac{h}{2}P)$ and $R \sin \psi_0 = \lambda h P$, where $P = -\mu + \sqrt{bQ^2 + \mu^2}$. The limit $x \to +1 - 0$ corresponds to a pure electric black hole while the limit $x \to -1 + 0$ corresponds to a pure magnetic one. To prove the relations (5.1) and (5.3) one should verify the inequality

$$-\psi_0 < \psi(x) < \psi_0 \tag{5.6}$$

for all $x \in (-1, 1)$. Let us suppose that equation (5.6) is not valid. Without loss of generality we put $\psi(x_*) \geqslant \psi_0$ for some x^*. Then using equations (5.5) and the smoothness of the function $\psi(x)$ we get that for some $x_1 \neq x_2$: $\psi(x_1) = \psi(x_2)$[2]. Hence for two different sets $(Q_1^2, Q_2^2)_1 \neq (Q_1^2, Q_2^2)_2$ we obtain the same coinciding sets: $(GM, Q_\varphi)_1 = (GM, Q_\varphi)_2$ and $(P_1, P_2)_1 = (P_1, P_2)_2$; see equations (4.1) and (4.2). But due to our conjecture the map $(Q_1^2, Q_2^2) \mapsto (P_1, P_2)$ is a bijective one. This implies $(P_1, P_2)_1 \neq (P_1, P_2)_2$. We get a contradiction which proves our proposition for $\varepsilon = 1$ and arbitrary $Q_1^2 + Q_2^2 > 0$. The proof of the right inequality in equation (5.2) and the bounds (5.3) for $\varepsilon = -1$ is quite an analogous one. The only difference here is the use of the parametrization

$$\sqrt{2}\,GM = R\cosh\psi, \qquad Q_\varphi = R\sinh\psi, \qquad R = \sqrt{Q^2 + 2\mu^2}, \tag{5.7}$$

instead of equation (5.4). □

The inequalities (5.1)–(5.3) imply the following bounds on mass and scalar charge, which are valid for all $\mu > 0$ and $h > 0$

$$\frac{1}{2}\sqrt{h\left(Q_1^2 + Q_2^2\right)} < GM, \tag{5.8}$$

for $\varepsilon = +1$ ($\lambda \neq 0$, $0 < h < 2$),

$$\sqrt{\frac{1}{2}\left(Q_1^2 + Q_2^2\right)} < GM, \tag{5.9}$$

for $\varepsilon = -1$ ($0 < \lambda^2 < \frac{1}{2}$, $h > 2$), and

$$|Q_\varphi| < |\lambda|\sqrt{h\left(Q_1^2 + Q_2^2\right)} \tag{5.10}$$

for both cases.

The bound (5.8) is in agreement with the bound (6.16) from [30] (BPS-like inequality), which was proved there by using certain spinor techniques.

It should be noted that in the pure electric case for $\varepsilon = +1$ Gibbons and Wells have proved another bound which relates $M, Q_\varphi, |Q_1|$ and λ [31]. An open question is whether such a bound could be extended somehow to the dyonic case.

Remark 4. For $h < 0$ the conjecture is not valid. This may be verified just by analysing the solutions with small charge Q_1 (or Q_2).

Remark 5. It should be noted that here we are dealing with a very special class of solutions with a phantom scalar field ($\varepsilon = -1$). Even in the limiting case $Q_2 = +0$ and $Q_1 \neq 0$ there exist several branches of phantom black hole solutions which are not covered by our analysis [32] (see also [33].)

The bounds (5.1)–(5.3) could be verified numerically by using the prescription which is described below. We outline some results of numerical calculations which are based on

[2] This may be readily proved by using the intermediate value theorem. This theorem states that if $f(x)$ is a continuous function on the interval $[a, b]$ then for any $d \in [f(a), f(b)]$, there is a point $c \in [a, b]$ such that $f(c) = d$. (Here for $f(a) > f(b)$, $[f(a), f(b)]$ is meant as $[f(b), f(a)]$.)

Table 1. Examples of numerical calculations for $\varepsilon = 1$; $\lambda = 0.5$; $\mu = 1$.

Q_1	Q_2	GM	Q_φ	Bounds on M and Q_φ
0.233313	0.165107	1.03923	0.0128309	true
0.233641	0.182372	1.0421	0.0100296	true
0.234003	0.199861	1.04528	0.00693443	true
0.234398	0.217596	1.04876	0.00353771	true
0.234828	0.235605	1.05256	−0.000169283	true
0.235293	0.253911	1.05669	−0.00419616	true

Table 2. Examples of numerical calculations for $\varepsilon = -1$; $\lambda = 0.5$; $\mu = 1$.

Q_1	Q_2	GM	Q_φ	Bounds on M and Q_φ
0.00408717	0.0111095	1.00007	0.0000533559	true
0.00408725	0.0122205	1.00008	0.0000663148	true
0.00408733	0.0133316	1.0001	0.0000805081	true
0.00408743	0.0144426	1.00011	0.0000959358	true
0.00408753	0.0155537	1.00013	0.000112598	true
0.00408764	0.0166649	1.00015	0.000130495	true

dynamical equations (2.11). We start with putting the boundary conditions on the horizon $z = 1$: $y_s(1)$, $s = 1, 2$.

Then for the first derivatives on the horizon $\left(\frac{dy_s}{dz}\right)_{|z=1} = y_s'(1)$ we obtain from (2.11)

$$y_s'(1) = bq_s^2 \exp\left(-2y_s(1) - ay_{\bar{s}}(1)\right), \tag{5.11}$$

$s = 1, 2$ ($\bar{s} = 2, 1$). For practical calculations we put $z = 1 - \delta$, where δ is small enough, say $\delta = 10^{-5}$, for initial values $y_s(1)$ about 1. This is necessary for a correct formulation of the Cauchy problem for equations (2.11).

Our strategy is the following one. For fixed λ and ε we start with the exact symmetric solution obeying $y_1(0) = y_2(0) = 0$, i.e. we put

$$y_1(1) = y_2(1) = b\ \ln(1 + p), \qquad q_1^2 = q_2^2 = p(p + 1). \tag{5.12}$$

See equation (3.10) and (3.11). Here $p = P/(2\mu) > 0$. Then we disturb relations (5.12) as follows

$$y_1(1) = b\ \ln(1 + p), \quad y_2(1) = kb\ \ln(1 + p), \quad q_1^2 = q_2^2 = p(p + 1), \tag{5.13}$$

where $k \neq 1$. We get a numerical solution with $y_1(0)$ and $y_2(0)$ not obviously equal to 0.

Now, we make a shift in our solutions

$$\bar{y}_s(z) = y_s(z) - y_s(0), \tag{5.14}$$

$s = 1, 2$.

The functions $\bar{y}_s(z)$ give us a new solution to the Toda-like equations (2.11) with rescaled charges

$$\bar{q}_s^2 = q_s^2 \exp\Big(2y_s(0) + ay_{\bar{s}}(0)\Big), \tag{5.15}$$

$s = 1, 2$. The crucial point here is that $\bar{y}_s(z)$ obeys the boundary conditions $\bar{y}_s(0) = 0$, $s = 1, 2$.

The asymptotical parameters P_s are extracted from the relations (4.4) (with y_s replaced by $\bar{y}_s$). The accuracy of calculations is controlled by equations (2.12) and (4.5).

Here we present certain examples of numerical data collected in tables 1 and 2. These data obey the bounds (5.1)–(5.3). Of course, these tables may be enlarged by adding (a vast number of) new lines.

6. Conclusions

In this paper a family of non-extremal black hole dyon solutions in a 4-dimensional model with a scalar field is presented. The scalar field is either an ordinary ($\varepsilon = +1$) or a ghost one ($\varepsilon = -1$). The solutions are defined up to two functions $H_1(R)$ and $H_2(R)$, which obey two differential equations of second order with boundary conditions imposed. For $\varepsilon = +1$ these equations are integrable for two cases when $\lambda^2 = 1/2$ or $\lambda^2 = 3/2$. There is also a special solution with coinciding electric and magnetic charges: $Q_1 = Q_2$, which is defined for all (admissible) ε and λ.

Here we have also calculated some physical parameters of the solutions: gravitational mass M, scalar charge Q_φ, Hawking temperature, black hole area entropy and post-Newtonian parameters β, γ. We have obtained a formula which relates M, Q_φ, dyon charges Q_1, Q_2, and the extremality parameter μ for all values of $\lambda \neq 0$. Remarkably, this formula does not contain λ. We have also shown that the product of the Hawking temperature and the Bekenstein–Hawking entropy does not depend upon ε, λ and the moduli functions of the solutions $H_s(R)$ as well.

We have calculated the PPN parameters β and γ without knowledge of explicit formulas for $H_s(R)$. The only assumption used was that these functions are given (at least) by an asymptotical series in $1/R$ in the vicinity of the zero point. We have found that $\gamma = 1$ and β do not depend upon λ and ε.

Here we have obtained bounds on the gravitational mass and scalar charge for $1 + 2\lambda^2\varepsilon > 0$ which are based on the conjecture (from section 5) on the parameters of solutions $P_1 = P_1(Q_1^2, Q_2^2)$, $P_2 = P_2(Q_1^2, Q_2^2)$. We have also presented several results of numerical calculations which support our bounds. A rigorous proof of this conjecture may be the subject of a separate publication as well as a detailed consideration of the case $\lambda^2 > 1/2$, $\varepsilon = -1$. For $\varepsilon = +1$ we have also deduced from our conjecture the well-known (unsaturated) lower bound on mass, which was obtained earlier by Gibbons *et al* [30] by using certain spinor techniques (just as in the well-known Nester–Witten approach).

An open question here is to find some physical (e.g. astrophysical) applications of the dyonic black hole solutions. Here one may consider a possible description of the black hole which is 'located' at the Galactic Center. Recently, it was shown that a near extremal Reissner–Nordström black hole provides a better fit of recent observational data for the black hole at the Galactic Center in comparison with the Schwarzschild black hole, see [34] and references therein. Dilatonic dyon black hole solutions (with certain scalar charge) may be used for a search of the best fit of the observational data for the black hole at the Galactic Center. For such research the thermodynamical calculations may be of relevance, e.g. due to possible analysis of the black hole stability, seach of bounds on variations of physical parameters, etc.

Acknowledgments

The paper was supported by the program of KazNU (Almaty) during the year 2014. VD acknowledges support from grant no. 0263/PCF–14 in fundamental research in natural sciences by the Ministry of Education and Science of Kazakhstan.

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