Energy conditions for a T^2 wormhole at the center

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Within general relativity, we determine the conditions needed for the existence of a toroidal T^2 wormhole. For this purpose, we employ the requirements of the positiveness of the second derivatives of the relevant components of the metric, which describe an increase in the linear sizes (or the area) of the cross section of the throat. The corresponding inequalities for the central energy density and pressures of the matter and for the metric are obtained.

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I. INTRODUCTION

The study of wormholes has a long history in General Relativity and in generalized theories of gravity (see, e.g., Refs. [1,2]). The nontrivial topology of wormhole solutions, where different regions of spacetime are connected via a throat, requires the presence of exotic matter in General Relativity [2–9], while in generalized theories of gravity, the gravitational interaction itself may provide effective stress-energy tensors, that allow for the violation of the energy conditions [10–18].

Most previous studies of wormholes have considered throats of spherical topology. In the case of static spherically symmetric wormholes, such a throat is simply given by a sphere and represents a minimal surface of the spacetime. When the throat is rotating, its geometry changes, since it becomes deformed due to the rotation [19–22], while its topology remains unchanged.

It appears interesting to consider also other throat topologies [1,2]. For instance, cylindrical wormholes possessing a topology $S^1 \times I$ have been investigated in [23–25]. [Here, the S^1 represents a circle in the (x, y) plane, while I corresponds to an interval on the z axis.] However, a particularly attractive throat topology is represented by a torus $T^2 = S^1 \times S^1$. But so far, such toroidal wormholes have been addressed only briefly in the literature [26,27].

Within General Relativity, the derivation of solutions describing a toroidal T^2 wormhole is an extremely complicated problem. The point is that the equations describing such a wormhole are systems of partial differential equations. This can already be seen from the fact that in toroidal coordinates [see Eq. (10)] the flat Minkowski spacetime metric depends on two coordinates. To obtain such solutions, it is necessary (a) to assign boundary conditions at the throat and at infinity, (b) to determine the properties of the matter needed to construct a T^2 wormhole, (c) and finally to obtain solutions of the partial differential equations (the Einstein-matter equations) subject to these boundary conditions. Here, asymptotically flat solutions are evidently of most interest.

Consequently, we expect that the problem of obtaining solutions describing toroidal T^2 wormholes should be split into several stages: (a) studying the properties of the matter needed to obtain toroidal wormholes, (b) investigating the asymptotic behavior of the solutions for T^2 wormholes, and (c) obtaining and solving the set of differential equations describing such wormholes subject to the appropriate boundary conditions. Presumably, these solutions should be sought numerically.

The present paper is a continuation of the study performed in Ref. [27], where we have obtained and studied a toroidal thin-shell wormhole. Here, we analyze the conditions imposed on the matter needed for the existence of a toroidal T^2 wormhole. To do this, we write down the Einstein-matter equations at the throat and assign the necessary geometric conditions providing the existence of a throat. For these conditions, we take the condition of the positiveness of the second derivatives of the relevant components of the metric, which describe an increase in the linear sizes (or the area) of the cross section of the throat.

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II. CONDITIONS AT THE THROAT FOR A S^2 WORMHOLE

To begin with, let us recall the procedure of obtaining the energy conditions for a S^2 wormhole at the throat, in order to repeat it for a toroidal T^2 wormhole in the next section. Let us take the following metric for a S^2 wormhole:

$$ds^2 = A(r)dt^2 - dr^2 - B(r)(d\theta^2 + \sin^2\theta d\varphi^2). \quad (1)$$

The Einstein equations are

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = \varkappa T^\mu_\nu, \quad (2)$$

where $\varkappa = 8\pi G$, and the energy-momentum tensor for the macroscopic matter is taken in the form

$$T^\mu_\nu = \mathrm{diag}(\epsilon, -p_r, -p_\theta, -p_\varphi). \quad (3)$$

In order for this energy-momentum tensor to be consistent with the spherically symmetric metric (1), it is necessary to take $p_\varphi = p_\theta$. Then, the Einstein equations (2) with the metric (1) yield the following equations:

$$\frac{A''}{A} + \frac{A'B'}{AB} - \frac{A'^2}{2A^2} = \varkappa(\epsilon + p_r + 2p_\theta), \quad (4)$$

$$\frac{B''}{B} - \frac{A'B'}{2AB} - \frac{B'^2}{2B^2} = -\varkappa(\epsilon + p_r). \quad (5)$$

Bearing in mind that the metric functions $A(r)$ and $B(r)$ must be even, we have the following expressions for the second derivatives at the throat:

$$\frac{A''_0}{A_0} = \varkappa[\epsilon_0 + (p_r)_0 + 2(p_\theta)_0], \quad (6)$$

$$\frac{B''_0}{B_0} = -\varkappa[\epsilon_0 + (p_r)_0]. \quad (7)$$

Here, the index 0 indicates that the value of a quantity is taken at the throat. For a spherically symmetric wormhole, the components of the metric $g_{\theta\theta} = -B$ and $g_{\varphi\varphi} = -B\sin^2\theta$ should have a minimum at the throat. Thus, the geometry of the wormhole throat yields

$$\epsilon_0 + (p_r)_0 < 0. \quad (8)$$

This implies the violation of the null energy condition everywhere at the throat.

III. CONDITIONS AT THE THROAT FOR A T^2 WORMHOLE

For a toroidal T^2 wormhole, we use the following metric:

$$ds^2 = f(\chi,\beta)dt^2 - l(\chi,\beta)d\chi^2 - g(\chi,\beta)d\beta^2 - \omega(\chi,\beta)d\varphi^2. \quad (9)$$

Here, t, χ, β, φ are toroidal coordinates, which, in a flat spacetime, describe the Minkowski metric as follows,

$$ds^2 = dt^2 - \left(\frac{a}{\cosh\alpha - \cos\beta}\right)^2 (d\alpha^2 + d\beta^2 + \sinh^2\alpha d\varphi^2), \quad (10)$$

where a is some parameter and the coordinate χ in (9) is related to the coordinate α from (10) as $\alpha = -\ln\chi$. Since only the $\binom{t}{t}$, $\binom{\beta}{\beta}$, $\binom{\varphi}{\varphi}$, $\binom{\chi}{\chi}$, and $\binom{\chi}{\beta}$ components of the Einstein equations (2) are nonvanishing, we choose the energy-momentum tensor as

$$T^\mu{}_\nu = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & -p_\chi & \frac{\sigma}{l} & 0 \\ 0 & \frac{\sigma}{g} & -p_\beta & 0 \\ 0 & 0 & 0 & -p_\varphi \end{pmatrix}, \quad (11)$$

where ϵ represents the energy density of the matter; p_χ, p_β, and p_φ are the components of its pressure; and σ is the density of momentum flux. This leads to the following equations for a toroidal T^2 wormhole:

$$-\frac{\omega_{,\chi,\chi}}{\omega} - \frac{l}{g}\frac{\omega_{,\beta,\beta}}{\omega} - \frac{g_{,\chi,\chi}}{g} - \frac{l_{,\beta,\beta}}{g} + \frac{g_{,\beta}l_{,\beta}}{2g^2} + \frac{l}{2}\frac{g_{,\beta}\omega_{,\beta}}{g^2\omega} + \frac{g^2_{,\chi}}{2g^2} + \frac{g_{,\chi}l_{,\chi}}{2lg} - \frac{g_{,\chi}\omega_{,\chi}}{2g\omega} + \frac{l^2_{,\beta}}{2lg} - \frac{l_{,\beta}\omega_{,\beta}}{2g\omega} + \frac{l_{,\chi}\omega_{,\chi}}{2l\omega} + \frac{l}{2g}\frac{\omega^2_{,\beta}}{\omega^2} + \frac{\omega^2_{,\chi}}{2\omega^2} = 2\varkappa l\epsilon, \quad (12)$$

$$-\frac{\omega_{,\beta,\beta}}{\omega} - \frac{f_{,\beta,\beta}}{f} + \frac{g_{,\beta}f_{,\beta}}{2gf} + \frac{g_{,\beta}\omega_{,\beta}}{2g\omega} - \frac{g_{,\chi}f_{,\chi}}{2lf} - \frac{g_{,\chi}\omega_{,\chi}}{2l\omega} + \frac{f^2_{,\beta}}{2f^2} - \frac{f_{,\beta}\omega_{,\beta}}{2f\omega} - \frac{g}{2l}\frac{f_{,\chi}\omega_{,\chi}}{f\omega} + \frac{\omega^2_{,\beta}}{2\omega^2} = -2\varkappa g p_\chi, \quad (13)$$

$$\frac{f_{,\beta,\chi}}{f} + \frac{\omega_{,\beta,\chi}}{\omega} - \frac{g_{,\chi}f_{,\beta}}{2gf} - \frac{g_{,\chi}\omega_{,\beta}}{2g\omega} - \frac{l_{,\beta}f_{,\chi}}{2lf} - \frac{l_{,\beta}\omega_{,\chi}}{2l\omega} - \frac{f_{,\chi}f_{,\beta}}{2f^2} - \frac{\omega_{,\chi}\omega_{,\beta}}{2\omega^2} = 2\varkappa\sigma, \quad (14)$$

$$-\frac{f_{,\chi\chi}}{f}-\frac{\omega_{,\chi\chi}}{\omega}-\frac{l_{,\beta}f_{,\beta}}{2gf}-\frac{l_{,\beta}\omega_{,\beta}}{2g\omega}-\frac{l}{2g}\frac{f_{,\beta}\omega_{,\beta}}{f\omega}+\frac{l_{,\chi}f_{,\chi}}{2lf}+\frac{l_{,\chi}\omega_{,\chi}}{2l\omega}+\frac{f_{,\chi}^2}{2f^2}-\frac{f_{,\chi}\omega_{,\chi}}{2f\omega}+\frac{\omega_{,\chi}^2}{2\omega^2}=-2\varkappa l p_\beta, \tag{15}$$

$$-\frac{f_{,\chi\chi}}{f}-\frac{g_{,\chi\chi}}{g}-\frac{l}{g}\frac{f_{,\beta\beta}}{f}-\frac{l_{,\beta\beta}}{g}+\frac{g_{,\beta}l_{,\beta}}{2g^2}+\frac{l}{2}\frac{g_{,\beta}f_{,\beta}}{gf}+\frac{g_{,\chi}^2}{2g^2}+\frac{g_{,\chi}l_{,\chi}}{2gl}-\frac{g_{,\chi}f_{,\chi}}{2gf}+\frac{l_{,\beta}^2}{2lg}-\frac{l_{,\beta}f_{,\beta}}{2gf}+\frac{l}{2g}\frac{f_{,\beta}^2}{f^2}+\frac{l_{,\chi}f_{,\chi}}{2lf}+\frac{f_{,\chi}^2}{2f^2}=-2\varkappa l p_\varphi. \tag{16}$$

Notice here that these equations can be solved in the case when the metric functions do not depend on the coordinate β. Such solutions will describe domain walls. In this case, the coordinates β, φ are the coordinates along the domain wall, and $-\infty < \beta, \varphi < +\infty$. If one rolls the domain wall into a torus, i.e., identifies $\beta \to \beta + n\beta_0$ and $\varphi \to \varphi + m\varphi_0$, where n, m are integers, one will have a toroidal T^2 wormhole, but it will not be asymptotically flat.

To determine the requirements imposed on the energy-momentum tensor of the matter supporting the wormhole, we, analogously to Sec. II, write down the $\binom{t}{t}$, $\binom{\beta}{\beta}$, and $\binom{\varphi}{\varphi}$ components of the Einstein equations at the throat, i.e., at $\chi = 0$, solving them with respect to higher-order derivatives $f_{,\chi\chi}$, $g_{,\chi\chi}$, and $\omega_{,\chi\chi}$:

$$\begin{aligned}(f_{,\chi\chi})_0 &= \varkappa f_0 l_0[\epsilon_0+(p_\beta)_0+(p_\varphi)_0]+\frac{f_0 l_0}{2\omega_0 g_0}(\omega_{,\beta\beta})_0-\frac{f_0 l_0}{4g_0}\frac{(\omega_{,\beta}^2)_0}{\omega_0^2}-\frac{f_0 l_0}{4g_0}\frac{(\omega_{,\beta})_0(g_{,\beta})_0}{\omega_0 g_0}-l_0\frac{(f_{,\beta})_0(\omega_{,\beta})_0}{4\omega_0 g_0}\\ &\quad+l_0\frac{(g_{,\beta})_0(f_{,\beta})_0}{4g_0^2}-\frac{(l_{,\beta})_0(f_{,\beta})_0}{2g_0}-l_0\frac{(f_{,\beta\beta})_0}{2g_0}+l_0\frac{(f_{,\beta}^2)_0}{4f_0 g_0},\end{aligned} \tag{17}$$

$$\begin{aligned}(g_{,\chi\chi})_0 &= \varkappa g_0 l_0[-\epsilon_0-(p_\beta)_0+(p_\varphi)_0]-(l_{,\beta\beta})_0-l_0\frac{(\omega_{,\beta\beta})_0}{2\omega_0}+l_0\frac{(\omega_{,\beta}^2)_0}{4\omega_0^2}+l_0\frac{(\omega_{,\beta})_0(g_{,\beta})_0}{4g_0\omega_0}+\frac{(g_{,\beta})_0(l_{,\beta})_0}{2g_0}\\ &\quad+\frac{(l_{,\beta}^2)_0}{2l_0}+l_0\frac{(f_{,\beta})_0(\omega_{,\beta})_0}{4\omega_0 f_0}+l_0\frac{(g_{,\beta})_0(f_{,\beta})_0}{4g_0 f_0}-l_0\frac{(f_{,\beta\beta})_0}{2f_0}+l_0\frac{(f_{,\beta}^2)_0}{4f_0^2},\end{aligned} \tag{18}$$

$$\begin{aligned}(\omega_{,\chi\chi})_0 &= \varkappa\omega_0 l_0[-\epsilon_0+(p_\beta)_0-(p_\varphi)_0]-\frac{l_0}{2g_0}(\omega_{,\beta\beta})_0+\frac{l_0}{4g_0\omega_0}(\omega_{,\beta}^2)_0+\frac{l_0}{4g_0^2}(\omega_{,\beta})_0(g_{,\beta})_0-\frac{(\omega_{,\beta})_0(l_{,\beta})_0}{2g_0}\\ &\quad-\frac{l_0}{4f_0 g_0}(f_{,\beta})_0(\omega_{,\beta})_0-\frac{\omega_0 l_0}{4f_0 g_0^2}(g_{,\beta})_0(f_{,\beta})_0+\frac{\omega_0 l_0}{2f_0 g_0}(f_{,\beta\beta})_0-\frac{\omega_0 l_0}{4f_0^2 g_0}(f_{,\beta}^2)_0.\end{aligned} \tag{19}$$

Here, we have taken into account that all functions are even,

$$\frac{\partial}{\partial\chi}[f(\chi,\beta), l(\chi,\beta), g(\chi,\beta), \omega(\chi,\beta)]|_{\chi=0}=0;$$

i.e., the wormhole is symmetric. The $\binom{\chi}{\chi}$ component of the Einstein equations is

$$\begin{aligned}&\frac{1}{2g_0}\left[\frac{(f_{,\beta\beta})_0}{f_0}+\frac{(\omega_{,\beta\beta})_0}{\omega_0}-\frac{(\omega_{,\beta}^2)_0}{2\omega_0^2}-\frac{(\omega_{,\beta})_0(g_{,\beta})_0}{2g_0\omega_0}\right.\\ &\quad\left.+\frac{(\omega_{,\beta})_0(f_{,\beta})_0}{2\omega_0 f_0}-\frac{(g_{,\beta})_0(f_{,\beta})_0}{2g_0 f_0}-\frac{(f_{,\beta}^2)_0}{2f_0^2}\right]=\varkappa(p_\chi)_0.\end{aligned} \tag{20}$$

The component $\binom{\chi}{\beta}$ and the equation

$$T^\mu{}_{\nu;\mu}=0 \tag{21}$$

for $\nu=\chi$ are satisfied when

$$\sigma_0(\beta)=\sigma(\chi=0,\beta)=0. \tag{22}$$

Equation (21) is satisfied for $\nu = t$, φ, and when $\nu=\beta$, it has the following form:

$$\begin{aligned}&\frac{(l_{,\beta})_0}{l_0}(p_\chi)_0+\left[\frac{(f_{,\beta})_0}{f_0}-\frac{(l_{,\beta})_0}{l_0}-\frac{(\omega_{,\beta})_0}{\omega_0}\right](p_\beta)_0\\ &\quad-2(p_{\chi,\beta})_0-\frac{(f_{,\beta})_0}{f_0}\epsilon_0+\frac{(\omega_{,\beta})_0}{\omega_0}(p_\varphi)_0=0.\end{aligned} \tag{23}$$

As usual, we assume that the necessary condition for a wormhole to exist (in the present case, a T^2 wormhole) is the presence of minima of the metric functions $g(\chi,\beta)=g_{\beta\beta}$ and $\omega(\chi,\beta)=g_{\varphi\varphi}$:

$$\left.\frac{\partial^2 g}{\partial\chi^2}\right|_{\chi=0} > 0, \qquad \left.\frac{\partial^2 \omega}{\partial\chi^2}\right|_{\chi=0} > 0. \tag{24}$$

Note that, instead of the inequalities (24), in Refs. [23,24], the authors discuss a necessary condition for the existence of a wormhole according to which a throat has a minimum of its two-dimensional space cross section. In our case, this corresponds to a minimum of the product $g \times \omega$, to yield

$$\left.\left(\frac{1}{g}\frac{\partial^2 g}{\partial\chi^2}\right)\right|_{\chi=0} + \left.\left(\frac{1}{\omega}\frac{\partial^2 \omega}{\partial\chi^2}\right)\right|_{\chi=0} > 0. \tag{25}$$

Here, we have taken into account the condition that the area of throat has a minimum, i.e., $\partial(g\omega)/\partial\chi = 0$.

Here, we will consider the inequalities (24), which are more strict than (25). This ensures that the cross section of a T^2 wormhole increases along both radii, when one moves away from the throat. In other words, the lengths of the circles $2\pi\sqrt{g}$ and $2\pi\sqrt{\omega}$ will increase, when one moves away from the throat. On the other hand, in the case when the inequality (25) is satisfied, the area of the cross section will increase, but the length of one of the circles may decrease, while the length of the other one increases.

Thus, Eq. (24) yields the following requirements imposed on the energy density and pressures of the matter needed to create a toroidal T^2 wormhole [they follow from Eqs. (18) and (19), respectively]:

$$\varkappa[\epsilon_0 + (p_\beta)_0 - (p_\varphi)_0] < \frac{1}{2g_0}\left\{\left[-\frac{(\omega_{,\beta,\beta})_0}{\omega_0} + \frac{(\omega^2_{,\beta})_0}{2\omega_0^2} + \frac{(\omega_{,\beta})_0(g_{,\beta})_0}{2g_0\omega_0}\right] + \left[-\frac{(f_{,\beta,\beta})_0}{f_0} + \frac{(f^2_{,\beta})_0}{2f_0^2} + \frac{(f_{,\beta})_0(\omega_{,\beta})_0}{2\omega_0 f_0} + \frac{(g_{,\beta})_0(f_{,\beta})_0}{2g_0 f_0}\right] - 2\frac{(l_{,\beta,\beta})_0}{l_0} + \frac{(g_{,\beta})_0(l_{,\beta})_0}{g_0 l_0} + \frac{(l^2_{,\beta})_0}{l_0^2}\right\}, \tag{26}$$

$$\varkappa[\epsilon_0 - (p_\beta)_0 + (p_\varphi)_0] < \frac{1}{2g_0}\left\{\left[-\frac{(\omega_{,\beta,\beta})_0}{\omega_0} + \frac{(\omega^2_{,\beta})_0}{2\omega_0^2} + \frac{(\omega_{,\beta})_0(g_{,\beta})_0}{2g_0\omega_0}\right] - \left[-\frac{(f_{,\beta,\beta})_0}{f_0} + \frac{(f^2_{,\beta})_0}{2f_0^2} + \frac{(f_{,\beta})_0(\omega_{,\beta})_0}{2\omega_0 f_0} + \frac{(g_{,\beta})_0(f_{,\beta})_0}{2g_0 f_0}\right] - \frac{(\omega_{,\beta})_0(l_{,\beta})_0}{\omega_0 l_0}\right\}. \tag{27}$$

Taking into account Eq. (20), the inequalities (26) and (27) can be rewritten in a simpler form to give

$$\varkappa[\epsilon_0 + (p_\chi)_0 + (p_\beta)_0 - (p_\varphi)_0] < \frac{1}{g_0}\left[-\frac{(l_{,\beta,\beta})_0}{l_0} + \frac{(l^2_{,\beta})_0}{2l_0^2} + \frac{(g_{,\beta})_0(l_{,\beta})_0}{2g_0 l_0} + \frac{(f_{,\beta})_0(\omega_{,\beta})_0}{2\omega_0 f_0}\right], \tag{28}$$

$$\varkappa[\epsilon_0 + (p_\chi)_0 - (p_\beta)_0 + (p_\varphi)_0] < \frac{1}{g_0}\left[\frac{(f_{,\beta,\beta})_0}{f_0} - \frac{(f^2_{,\beta})_0}{2f_0^2} - \frac{(g_{,\beta})_0(f_{,\beta})_0}{2g_0 f_0} - \frac{(\omega_{,\beta})_0(l_{,\beta})_0}{2\omega_0 l_0}\right]. \tag{29}$$

The inequality (25) describing a minimum of the area of the throat can be rewritten in the form

$$\varkappa\epsilon_0 < \frac{1}{2g_0}\left[-\frac{(l_{,\beta,\beta})_0}{l_0} - \frac{(\omega_{,\beta,\beta})_0}{\omega_0} + \frac{(\omega^2_{,\beta})_0}{2\omega_0^2} + \frac{(l^2_{,\beta})_0}{2l_0^2} + \frac{(\omega_{,\beta})_0(g_{,\beta})_0}{2\omega_0 g_0} - \frac{(\omega_{,\beta})_0(l_{,\beta})_0}{2\omega_0 l_0} + \frac{(g_{,\beta})_0(l_{,\beta})_0}{2g_0 l_0}\right]. \tag{30}$$

IV. ANALYSIS OF THE ENERGY CONDITIONS FOR A T^2 WORMHOLE

In constructing wormhole solutions, it is of great interest to study the question of violation of the energy conditions at the throat: to what extent is such a violation necessary for the throat to exist? For a static S^2 wormhole, the answer is positive: the null energy condition is violated everywhere at the throat. In this section, we consider some particular conditions of violation (or nonviolation) of the energy conditions for a T^2 wormhole.

For convenience of performing calculations, let us introduce new functions

$$f(\chi,\beta) = e^{F(\chi,\beta)}, \qquad g(\chi,\beta) = e^{G(\chi,\beta)},$$
$$l(\chi,\beta) = e^{L(\chi,\beta)}, \qquad \omega(\chi,\beta) = e^{\Omega(\chi,\beta)}. \tag{31}$$

Using them, we analyze the conditions (26) and (27) which are necessary for the existence of minima of the metric components $g_{\beta\beta}$ and $g_{\varphi\varphi}$. The inequalities (26) and (27) can be reduced to a more symmetric form if we set

$$L_{,\beta}(\chi = 0, \beta) = 0. \tag{32}$$

This condition means that the metric function $g_{\chi\chi}$ is constant at the center of the wormhole. Then, the first terms on the left- and right-hand sides of the inequalities (26) and (27) are the same, and the second terms have different signs.

To satisfy these inequalities, one can consider the following particular case when the second terms on the left-hand sides are equal to the second terms on the right-hand sides:

$$\varkappa[(p_\beta)_0 - (p_\varphi)_0] = \frac{e^{-G}}{2}\left[-(F_{,\beta,\beta})_0 - \frac{(F^2_{,\beta})_0}{2} + \frac{(F_{,\beta})_0(\Omega_{,\beta})_0}{2} + \frac{(F_{,\beta})_0(G_{,\beta})_0}{2}\right]. \tag{33}$$

In the following, we will study the consequences for this particular case. The inequalities (26) and (27) are now identical and read

$$\varkappa\epsilon_0 < \frac{e^{-G}}{2}\left[-(\Omega_{,\beta,\beta})_0 - \frac{(\Omega^2_{,\beta})_0}{2} + \frac{(\Omega_{,\beta})_0(G_{,\beta})_0}{2}\right], \tag{34}$$

Thus, we have Eqs. (20), (23), and (33) and inequality (34).

Equations (20), (23), and (33) can be solved with respect to the pressures, $(p_\chi)_0$, $(p_\beta)_0$, and $(p_\varphi)_0$,

$$(p_\chi)_0 = \frac{e^{-G_0}}{2\varkappa}\left[(F_{,\beta,\beta})_0 + \frac{(F^2_{,\beta})_0}{2} + (\Omega_{,\beta,\beta})_0 + \frac{(\Omega^2_{,\beta})_0}{2} + \frac{(F_{,\beta})_0(\Omega_{,\beta})_0}{2} - \frac{(F_{,\beta})_0(G_{,\beta})_0}{2} - \frac{(G_{,\beta})_0(\Omega_{,\beta})_0}{2}\right], \tag{35}$$

$$(p_\beta)_0 = \epsilon_0 + 2\frac{(p_{\chi,\beta})_0}{F_{,\beta}} + \frac{e^{-G_0}}{2\varkappa}\left\{-\frac{(F_{,\beta,\beta})_0(\Omega_{,\beta})_0}{(F_{,\beta})_0} + \frac{(\Omega_{,\beta})_0^2}{2} + \frac{(\Omega_{,\beta})_0}{2}[-(F_{,\beta})_0 + (G_{,\beta})_0]\right\}, \tag{36}$$

$$(p_\varphi)_0 = \epsilon_0 + 2\frac{(p_{\chi,\beta})_0}{F_{,\beta}} + \frac{e^{-G_0}}{2\varkappa}\left\{(F_{,\beta,\beta})_0\left[1 - \frac{(\Omega_{,\beta})_0}{(F_{,\beta})_0}\right] + \frac{(F_{,\beta})_0^2}{2} - \frac{(F_{,\beta})_0(G_{,\beta})_0}{2} + \frac{(\Omega_{,\beta})_0^2}{2} + \frac{(\Omega_{,\beta})_0}{2}[-2(F_{,\beta})_0 + (G_{,\beta})_0]\right\}. \tag{37}$$

Let us now analyze the energy conditions.

A. Null energy condition

In general, the null energy condition asserts that for any null vector k_μ

$$T_{\mu\nu}k^\mu k^\nu \geq 0, \tag{38}$$

or in terms of the principal pressures p_i,

$$\epsilon_0 + p_i \geq 0, \quad i = \chi, \beta, \varphi. \tag{39}$$

In our particular case, Eq. (33), we have the following expressions for the left-hand sides of Eq. (39):

$$\epsilon_0 + (p_\chi)_0 = \epsilon_0 + \frac{e^{-G_0}}{2\varkappa}\left[(F_{,\beta,\beta})_0 + \frac{(F^2_{,\beta})_0}{2} + (\Omega_{,\beta,\beta})_0 + \frac{(\Omega^2_{,\beta})_0}{2} + \frac{(F_{,\beta})_0(\Omega_{,\beta})_0}{2} - \frac{(F_{,\beta})_0(G_{,\beta})_0}{2} - \frac{(G_{,\beta})_0(\Omega_{,\beta})_0}{2}\right], \tag{40}$$

$$\epsilon_0 + (p_\beta)_0 = 2\epsilon_0 + 2\frac{(p_{\chi,\beta})_0}{F_{,\beta}} + \frac{e^{-G_0}}{2\varkappa}\left\{-\frac{(F_{,\beta,\beta})_0(\Omega_{,\beta})_0}{(F_{,\beta})_0} + \frac{(\Omega_{,\beta})_0^2}{2} + \frac{(\Omega_{,\beta})_0}{2}[-(F_{,\beta})_0 + (G_{,\beta})_0]\right\}, \tag{41}$$

$$\begin{aligned}\epsilon_0 + (p_\varphi)_0 = 2\epsilon_0 + 2\frac{(p_{\chi,\beta})_0}{F_{,\beta}} + \frac{e^{-G_0}}{2\varkappa}\bigg\{&(F_{,\beta,\beta})_0\left[1 - \frac{(\Omega_{,\beta})_0}{(F_{,\beta})_0}\right] + \frac{(F_{,\beta})_0^2}{2} - \frac{(F_{,\beta})_0(G_{,\beta})_0}{2} + \frac{(\Omega_{,\beta})_0^2}{2}\\ &+ \frac{(\Omega_{,\beta})_0}{2}[-2(F_{,\beta})_0 + (G_{,\beta})_0]\bigg\}.\end{aligned} \tag{42}$$

B. Weak energy condition

The weak energy condition asserts in general that for any timelike vector V_μ

$$T_{\mu\nu}V^\mu V^\nu \geq 0. \tag{43}$$

In our case, this gives

$$\epsilon_0 \geq 0, \tag{44}$$

$$\epsilon_0 + p_i \geq 0, \quad i = \chi, \beta, \varphi. \tag{45}$$

The energy density ϵ_0 satisfies the inequality (34), and it can be positive if the right-hand side of this inequality will be positive in a whole range $-\pi \leq \beta \leq \pi$. In the particular case of Eq. (33), the left-hand sides of the inequality (45) have the form (40)–(42).

C. Strong energy condition

The strong energy condition asserts in general that for any timelike vector V_μ

$$\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)V^\mu V^\nu \geq 0. \tag{46}$$

In our case, we then have

$$\epsilon_0 + (p_i)_0 \geq 0, \quad i = \chi, \beta, \varphi, \tag{47}$$

$$\epsilon_0 + \sum_i (p_i)_0 \geq 0. \tag{48}$$

The energy density satisfies the inequality (34), and it can be positive if the right-hand side of this inequality will be positive in a whole range $-\pi \leq \beta \leq \pi$. In the particular case of Eq. (33) and taking into account Eqs. (40)–(42), the left-hand side of the inequality (48) takes the form

$$\epsilon_0 + (p_\chi)_0 + (p_\beta)_0 + (p_\varphi)_0 = 3\epsilon_0 + 4\frac{(p_{\chi,\beta})_0}{F_{,\beta}} + \frac{e^{-G_0}}{2\varkappa}\left\{(\Omega_{,\beta,\beta})_0 + 2(F_{,\beta,\beta})_0\left[1 - \frac{(\Omega_{,\beta})_0}{(F_{,\beta})_0}\right] + \frac{3(\Omega_{,\beta})_0^2}{2} + \frac{(\Omega_{,\beta})_0(G_{,\beta})_0}{2} - (\Omega_{,\beta})_0(F_{,\beta})_0 + (F_{,\beta})_0^2 - (F_{,\beta})_0(G_{,\beta})_0\right\}. \tag{49}$$

Summarizing, we see that to construct a toroidal T^2 throat there are some necessary conditions to be imposed on the matter supporting the wormhole. Of course, this does not ensure the existence of an asymptotically flat wormhole; to obtain such a wormhole, it is necessary to assign also asymptotic boundary conditions providing the asymptotic flatness of spacetime.

V. PARTICULAR CASES

We see that, even if $L_{,\beta} = 0$, the inequality (34) and the expressions for the pressures (35)–(37) are too cumbersome to perform the analysis of the conditions imposed on the matter, which are necessary for the existence of a toroidal T^2 wormhole. Therefore, in this section, we consider some particular cases permitting a simplification of the equations.

A. Particular case with the positive right-hand side of the inequality (34)

Consider the conditions providing the positiveness of the right-hand side of the inequality (34). This assumes that the energy density $\epsilon_0 > 0$. For this purpose, we take

$$(\Omega(\beta))_0 = -\cos\beta, \tag{50}$$

$$(G(\beta))_0 = -3\cos\beta + \log\sin^2\beta. \tag{51}$$

In this case, the inequality (34) yields

$$\varkappa\epsilon_0 < \frac{e^{3\cos\beta}}{2}. \tag{52}$$

For comparison, we have the following inequality from Ref. [28],

$$\rho_0 \leq \frac{1}{16\pi G}{}^{(2)}R_0, \tag{53}$$

where ${}^{(2)}R_0$ is the scalar curvature of the throat (which is a torus); here, the speed of light is $c = 1$, and consequently $\epsilon_0 = \rho_0$. Integrating over the torus and taking into account the Gauss-Bonnet theorem, we obtain

$$\int_{\text{throat}} \epsilon_0 dS \leq 0. \tag{54}$$

This means that ϵ_0 cannot be positive everywhere. However, we cannot know beforehand whether in such a case there exists a global, asymptotically flat solution.

In turn, the expressions for the pressures are as follows:

$$(p_\chi)_0 = \frac{e^{-G_0}}{2\varkappa}\left\{(F_{,\beta,\beta})_0 + \frac{(F^2_{,\beta})_0}{2} - (F_{,\beta})_0[(\Omega_{,\beta})_0 - (G_{,\beta})_0]\right\} - \frac{e^{3\cos\beta}}{2\varkappa}, \tag{55}$$

$$(p_\beta)_0 = \epsilon_0 + 2\frac{(p_{\chi,\beta})_0}{F_{,\beta}} + \frac{e^{-G_0}}{2\varkappa}\left\{-\frac{(F_{,\beta\beta})_0(\Omega_{,\beta})_0}{(F_{,\beta})_0} + \frac{(\Omega_{,\beta})_0^2}{2} + \frac{(\Omega_{,\beta})_0}{2}[-(F_{,\beta})_0 + (G_{,\beta})_0]\right\}, \quad (56)$$

$$(p_\varphi)_0 = \epsilon_0 + 2\frac{(p_{\chi,\beta})_0}{F_{,\beta}} + \frac{e^{-G_0}}{2\varkappa}\left\{(F_{,\beta\beta})_0\left[1 - \frac{(\Omega_{,\beta})_0}{(F_{,\beta})_0}\right] + \frac{(F_{,\beta})_0^2}{2} - \frac{(F_{,\beta})_0(G_{,\beta})_0}{2} + \frac{(\Omega_{,\beta})_0^2}{2} + \frac{(\Omega_{,\beta})_0}{2}[-2(F_{,\beta})_0 + (G_{,\beta})_0]\right\}. \quad (57)$$

B. Particular case $L_{\beta} = F_{,\beta} = 0$

Consider now an even more simplified case, when $F_{,\beta}(\chi = 0, \beta) = 0$ as well. As one can see from Eq. (33), this assumes the equality of the tangential pressures at the throat,

$$(p_\beta)_0 = (p_\varphi)_0. \quad (58)$$

Next, the pressure $(p_\chi)_0$ is

$$(p_\chi)_0 = \frac{e^{-G_0}}{2\varkappa}\left[(\Omega_{,\beta\beta})_0 + \frac{(\Omega^2_{,\beta})_0}{2} - \frac{(G_{,\beta})_0(\Omega_{,\beta})_0}{2}\right]. \quad (59)$$

By comparing this expression to the inequality (34), we have the following inequality for the energy density ϵ_0 and the pressure $(p_\chi)_0$:

$$\epsilon_0 + (p_\chi)_0 < 0. \quad (60)$$

According to (39), this implies the violation of the null energy condition everywhere at the throat. At the same time, the pressures $(p_\beta)_0$ and $(p_\varphi)_0$ remain arbitrary, obeying only the condition (58).

VI. CONCLUSIONS

In order to obtain a toroidal T^2 wormhole, it is necessary to solve the following problems: (a) to assign boundary conditions at the throat, (b) to assign asymptotic boundary conditions at infinity, and (c) to obtain numerical solutions (since presumably it will be impossible to get analytical solutions because of the complexity of the partial differential Einstein equations).

Each of these problems is quite complicated. To solve problem a, it is necessary to analyze the conditions imposed on the matter supporting the wormhole: to what extent the violation of the energy conditions at the throat is needed or not. Perhaps, the violation of the energy conditions is necessary only on a part of the torus T^2 forming the throat.

To solve problem b, it is necessary to find an asymptotic analytical solution of the Einstein equations at infinity. The problem is that the spatial infinity is given by the conditions $\alpha = \beta = 0$ in the metric (10), and the analysis of this (coordinate) singularity is not simple. The behavior of the metric functions depends on the relation between α and β, when they approach zero.

Problem c consists in obtaining numerical solutions to the corresponding Einstein-matter equations. These equations will in general be partial differential equations with boundary conditions assigned at the throat and at infinity. It is clear that finding the numerical solution of such a set of equations represents a great challenge.

In the present paper, we have studied problem a. We have obtained the inequalities describing the conditions needed for a toroidal T^2 wormhole to exist, assuming that minima of all metric functions (or the area) are reached at $\chi = 0$ for all values of the angular coordinate β simultaneously. These conditions are geometric, and they define the requirements for the energy density, pressures, and the metric, providing a minimum of the linear sizes of the cross section of a toroidal wormhole at the throat. Physically, these inequalities describe the conditions for the matter supporting a toroidal T^2 wormhole. Here, we have obtained these conditions in a general form. These conditions have complicated and intricate form; therefore, in order to obtain more concrete results clarifying the physical situation, we have analyzed the derived inequalities in some special cases. In one particular case, we have shown that a T^2 throat may exist only when the null energy condition is violated everywhere at the throat. Notice that our results agree with Page's argumentation, which has been pointed out by the authors of Ref. [7] on p. 405, according to which the null energy condition must be violated not only for spherical wormholes but for nonspherical systems as well. The strict mathematical proof of this fact was given in Ref. [28].

The results obtained can be summarized as follows:

- (i) We have derived the Einstein equations for a toroidal T^2 wormhole.
- (ii) The necessary conditions for the T^2 wormhole to exist have been obtained (the restrictions on matter at the center of the wormhole).
- (iii) These conditions have been analyzed for the case when the metric function $g_{\chi\chi}$ is constant at the throat:
  - (1) In the first case, the energy density has constraints (52) and (54), and the pressures take certain values (55)–(57).
  - (2) In the second case, the null energy condition is violated, and the pressures take definite values.
- (iv) It is shown that the energy density and pressures are required to be functions of the angle β; otherwise, one has asymptotically nonflat wormholes. Such wormholes can be obtained from any domain-wall type solution by rolling the wall into a torus.

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