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Nonperturbative Quantization à La Heisenberg: Modified Gravities, Wheeler-DeWitt Equations, and Monopoles in QCD

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Abstract—For field theories in which no small parameter is available, we use Heisenberg's quantization procedure to propose a definition of nonperturbative quantum states in terms of the complete set of Green functions. We present the corresponding quantization schemes in the case of Einstein gravity and gauge theories. To illustrate the procedure of quantization, we show that: (1) modified theories of gravity appear as an effective approximation of nonperturbative quantum gravity; (2) the Wheeler-DeWitt equations appear as a sort of approximation of the quantization procedure á la Heisenberg, and (3) it is possible to carry out explicit nonperturbative calculations in quantum chromodynamics, and we obtain the energy spectrum of a quantum monopole and some thermodynamic quantities for a gas of noninteracting quantum monopoles.

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1. INTRODUCTION

In theoretical physics, it is now widely believed that the physical world has an intrinsic quantum nature. It should, therefore, be possible to develop an exact quantum description of all physical systems. However, this is not the case. Indeed, a quantum-mechanical description implies solving the Schrödinger equation for a wave function. Simple mechanical systems can be treated in this way, and idealized solutions can be found analytically. Nevertheless, more realistic physical systems imply a direct numerical solution of Schrödinger's equation, using a finite-element approximation for the operators involved in the corresponding differential equation. In general, this method is very ineffective due to a large number of resulting variables. It is, therefore, necessary to replace the physical system with an idealized model which admits an exact solution. Then, one tries to determine a perturbation series for the perturbation operator which is determined by the difference between the Hamiltonian of the physical system and the Hamiltonian of the idealized model. This approach can be applied only if there exist small parameters with respect to which the perturbation operator can be expanded. Most physical systems, however, do not permit the existence of such a small parameter that could be used in the framework of perturbation theory. Moreover, it can happen that the small parameter exists, but the perturbation series has a vanishing convergence radius. In all these cases, a nonperturbative approach is necessary in order to describe the quantum properties of the physical system.

An essential component of perturbative quantization is the Fock space of quantum states which is constructed as follows. The perturbative approach allows us to introduce the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$, respectively. Then, the vacuum state $|\text{vac}\rangle$ is introduced by means of the condition

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$\hat{a}|\text{vac}\rangle = 0$. Analogously, a quantum state with n particles is defined as $|n\rangle = (\hat{a}^\dagger)^n|\text{vac}\rangle$. The Fock space is understood as a sum of all $n-$particle states $|n\rangle$. We see that the concept of a particle is essential for introduce the Fock space. If there is no small parameter, the Fock space becomes inadequate, and a description of the physical system in terms of particles is problematic. This is an important fact that must be considered in the construction of any alternative nonperturbative approach to quantization.

The problem of handling physical systems with no small parameter has been treated in many different ways. For instance, direct numerical integration of the Schrödinger equation or the corresponding field equations, in which an auxiliary parameter (the step of integration) is introduced, has been used intensively. In this connection, lattice models are commonly used in quantum field theory; in this case, the lattice constant, which is an auxiliary nonphysical parameter, used to calculate functional integrals, determines the accuracy of the lattice approach. Numerical methods usually encounter technical problems due to the exponential growth of calculations. Moreover, physical systems with a large number of degrees of freedom lead to instabilities of the numerical approaches.

Nonnumerical approaches include the Hartree-Fock method, the procedures for approximating Hamiltonians, the density functional theory, etc. Most of these nonnumerical methods can solve the main problem of estimating the physical properties of the ground state. However, successive states cannot in general be calculated with the required accuracy, and the determination of the spectrum of the entire system is a difficult task in most of systems [1]. Some of these difficulties are overcome by the operator method which reduces all differential calculations to algebraic calculus with matrix elements of the operators. The eigenfunctions and eigenvalues of the Hamiltonian are easily calculated in the zeroth approximation, whereas successive approximations lead to convergent sequences, so that physical quantities can be estimated with any desirable accuracy [2]. In the context of supersymmetric quantum field theory, the orbifold equivalence and other methods have been applied to understand quark confinement in gauge theories. These investigations have also provided some progress towards formulation of a nonperturbative definition of the path integral in quantum field theory [3].

Although there have been several attempts to construct a consistent approach to nonperturbative quantization, and many interesting technical results have been obtained, no definite approach has been formulated so far. In view of this situation, we believe that it is convenient to explore alternative methods which could shed some light on the complexity of the problem. In this work, we follow this strategy and propose a definition of nonperturbative quantum states which can be used in a consistent manner to find certain characteristics of quantum systems. This new definition is based on the use of Green functions in Heisenberg's quantization procedure. The main idea is simple. Since the Green functions are solutions of the corresponding operator field equations, they should contain all quantum information about the physical system.

A major problem of modern theoretical physics is quantization of the gravitational field. We consider this important case here and present a review of the quantization scheme for Einstein gravity, in which quantum fluctuations and two-point quantum correlations of the metric are considered. We compare the Wheeler-DeWitt equations with the nonperturbative quantization scheme of Einstein gravity and show explicitly the conceptual differences of the two approaches. In addition, we study the case of an SU(3) gauge theory.

To exemplify our method, we will study in this work the thermodynamics of a quantum monopole and of a dilute monopole gas. The main difference between the thermodynamics of perturbatively quantized fields and the thermodynamics of nonperturbatively quantized fields is that in the first case a physical system can be enclosed in an arbitrary volume filled with particles — quanta (for example, photons). In the second case the fields can create self-supporting objects with fields exponentially decaying at infinity. These objects are protons, neutrons, nuclei, glueballs, etc. In the first case, calculation techniques for determining the partition function are well known. In the second case, the energy of such a physical system is practically concentrated in a restricted region of space without any walls. The linear size and volume of this region are determined by some field parameters.

In QCD lattice calculations, it was shown with great certainty that the QCD vacuum has a complex structure: it contains magnetic monopoles [5–8]. A great deal of works has been devoted to studies of monopoles in different aspects, including the problem of confinement in QCD, the problem of proton decay, in cosmology and astrophysics, etc. (see, e.g., an extensive literature on the subject in the book [9]). In the context of the present paper, we just make a brief mention of some of these works devoted to a monopole condensate and a dilute gas of monopoles. In [10, 11], the dual superconductivity of vacuum in SU(3) gauge theory is investigated by constructing a disorder parameter which signals monopole condensation. In those studies, the Abelian projection method was used, within which, as some believe, it is possible to get monopoles that are more relevant than others for confinement. Numerical studies of Abelian

monopoles in lattice gauge theory are presented in [12, 13]. In turn, a study of SU(2) dilute monopole gas was initiated by Polyakov's paper [14], where, in the framework of semiclassical quantization of SU(2) monopole solutions, the effect of Debye screening by a dilute gas of monopoles was demonstrated. Developing this idea, the dilute gas of monopoles was considered in different aspects (including the thermodynamic ones) in [15–21]. These studies include a consideration of magnetic monopoles in a fully nonperturbative way in lattice Monte Carlo simulations [21]. In turn, the authors of [19] come to the conclusion that even though the Abelian monopole gas in 3-dimensional SU(2) gluodynamics is not dilute, the dilute monopole gas approximation is adequately consistent with the measurements of monopole density and the Debye screening mass.

Consistent with all this, one of the purposes of the present paper is to demonstrate the fact that there is a possibility to study thermodynamics of a dilute gas of monopoles within the nonperturbative quantization à la Heisenberg. To do that, we first examine the thermodynamics of a single quantum monopole obtained by one of us earlier in [22], after which we compute the thermodynamic functions of the dilute gas of monopoles.

The physical system under investigation is a quantum condensate filled with quantum monopoles. The quantum monopoles are supported by those degrees of freedom of the SU(3) gauge field that belong to the SU(2) subgroup of the SU(3) gauge field, i.e., $\mathrm{SU}(2) \subset \mathrm{SU}(3)$. The quantum condensate is described by the coset degrees of freedom SU(3)/SU(2).

There is a crucial difference between these degrees of freedom: the SU(2) fields have a nonzero quantum average with quantum fluctuations around this average. The dispersion of these quantum fluctuations is approximately described as some constant determined from solving a nonlinear eigenvalue problem. To find quantum averages of the SU(2) field, we use the Yang-Mills equation corresponding to the SU(2) subgroup. In turn, the quantum condensate is supported by the coset space gauge fields, which have a zero quantum average and nonzero dispersion, and whose expectation value over all color and spacetime indices is described by the scalar field ϕ.

To describe the physical system under consideration, we employ the two-equation approximation of [22, 23], within which the Yang-Mills and scalar-field equations are solved as a nonlinear eigenvalue problem. Having such a solution, we calculate the energy spectrum of a quantum monopole that enables us to evaluate a partition function for a single quantum monopole and also to get all the corresponding thermodynamic functions.

In computing the partition function of the monopole gas, we take into account the internal structure of monopoles that results in the appearance of an extra term in the expression for the energy of the physical system under investigation. To simplify the difficult problem of determining the energy of the system of quantum monopoles, we assume that the motion of monopoles (and hence their kinetic energy) can be represented as the motion of pointlike particles.

This work is organized as follows. In Sec. 2, we use Heisenberg's quantization to propose a definition of nonperturbative quantum states in terms of Green functions. Some properties of this definition are also discussed. We present the scheme of quantization in the case of Einstein gravity and in the case of an SU(3) gauge field. In Sec. 4, in the framework of the two-equation approximation, we construct the energy spectra of a single monopole and of a flux tube which can connect a monopole and n antimonopole in the vacuum of QCD. Using these solutions, we evaluate numerically the partition function of a single monopole and calculate the corresponding thermodynamic quantities. In Sec. 5 we consider a nonrelativistic dilute gas of noninteracting monopoles, for which we find the partition function and derive the resulting thermodynamics. In Sec. 6 we demonstrate the relationship between one of free parameters of our system and the typical energy scale of QCD. Finally, in Sec. 7 we summarize the results.

2. NONPERTURBATIVE QUANTUM STATES

The absence of a small parameter in a physical system prevents it from being quantized perturbatively and, consequently, the Fock space from being constructed explicitly, although a formal mathematical definition is available as a special case of the Hilbert space. This means that alternative approaches might be considered in order to characterize nonperturbative quantum states. As an alternative approach, we will consider, in the following analysis, Heisenberg's quantization method [4]. To be more specific, consider a Lagrangian density functional $\mathcal{L}(\Phi^A)$ for a set of fields $\Phi^A(x)$. The variational principle leads to the field equations

$$D\mathcal{L} \equiv \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \Phi^A_{,\mu}}\right) - \frac{\partial \mathcal{L}}{\partial \Phi^A} = 0. \qquad (1)$$

According to Heisenberg's procedure, the first step towards nonperturbative quantization of a classical system consists in replacing the classical fields Φ^A with the corresponding operators $\hat{\Phi}^A$. Then, the quantum counterpart of the above field equations is

$$D\hat{\mathcal{L}} = 0, \quad \hat{\mathcal{L}} = \hat{\mathcal{L}}(\hat{\Phi}^A). \qquad (2)$$

In general, there is no method of solving this kind of operator equation. To avoid this difficulty, one can alternatively consider the average of the operator equation over all possible products of the field operator Φ^A, i.e., one considers the entire set of Green functions which are determined by means of the set of equations

$$\begin{gathered}\left\langle Q\left|T\left(D\hat{\mathcal{L}}\right)\right|Q\right\rangle=0,\\ \left\langle Q\left|T\left(\hat{\Phi}^A(x_1)D\hat{\mathcal{L}}\right)\right|Q\right\rangle=0,\\ \left\langle Q\left|T\left(\hat{\Phi}^{A_1}(x_1)\hat{\Phi}^{A_2}(x_2)D\hat{\mathcal{L}}\right)\right|Q\right\rangle=0,\\ \cdots=0,\\ \left\langle Q\left|T\left(\hat{\Phi}^{A_1}(x_1)\hat{\Phi}^{A_2}(x_2)\cdots\hat{\Phi}^{A_n}(x_n)D\hat{\mathcal{L}}\right)\right|Q\right\rangle=0,\\ \cdots=0. \end{gathered} \tag{3}$$

Here T is the time-ordering operator. This represents an infinite set of differential equations that must be solved for any particular physical system; if this can be done, we end up with an infinite set of Green functions which should contain all physical information about the field operators and the quantum states of the system. *It then follows that we can identify the set of nonperturbative quantum states with the set of Green functions*. This is the main observation that allows us to formally define nonperturbative quantum states in terms of Green functions.

We, therefore, propose the following nonperturbative quantization scheme.

(1) The quantum states are defined as a set of all Green functions that are special solutions of the corresponding operator equations. This means that the set of all nonperturbative quantum states represents a generalization of the Fock space.

(2) All physical information of the physical system is encoded in the Green functions, i.e., average values of all physical quantities, their dispersion, scattering amplitudes and so on.

(3) The properties of quantum operators are also defined in terms of the Green functions.

According to the above scheme, the Green functions determine quantum states and the properties of quantum operators and their commutation relations as well.

Of course, solving an infinite set of equations is not possible, unless we come up with a method able to recursively solve the equations or to truncate the equation series. In general, it is not an easy task. Nevertheless, the situation is not hopeless since examples can be found in which this procedure leads to consistent results. Consider, for instance, the case of a linear field theory. In this case, it is possible to calculate the propagator; then, all Green functions can be represented as polylinear combinations of the propagator. In other words, Heisenberg's nonperturbative approach is simply equivalent to canonical quantization of linear systems in terms of propagators. We thus see that our definition of nonperturbative quantum states in terms of Green functions is trivially realized in the case of linear field theories.

An important consequence of defining quantum states as above is that the classical limit, which is usually a difficult task in most nonperturbative approaches, can easily be handled. Indeed, if all Green functions can be represented as products of the field functions, then the system is classic. Consequently, to obtain the classical limit, it is necessary to represent the complete set of Green functions in terms of standard field functions. This can be considered as a nontrivial advantage of our definition of nonperturbative quantum states.

2.1. Modified Gravities

To illustrate the above ideas on nonperturbative quantization of strong interacting fields, we will now consider the case of quantum gravity. It should be mentioned that the approach we present in this section is by no means a complete approach to quantization of gravity. In fact, a very important aspect of such a theory should be discretization of space itself. This has been done with certain restrictions only in loop quantum gravity [28]. Here, we restrict ourselves to a description of the main equations which must be solved in order to understand, to some extent, the meaning of nonperturbative quantum gravity à la Heisenberg. The operator equation (2) can be written in the case of Einstein gravity as

$$\hat{G}_{\mu\nu}=\hat{R}_{\mu\nu}-\frac{1}{2}\hat{g}_{\mu\nu}\hat{R}=\varkappa\hat{T}_{\mu\nu}, \tag{4}$$

In addition, one should be able to consider the quantum nature of space (area and volume) and its interaction with the field. This is one of the issues that currently makes quantum gravity an unattainable theory. One can, however, consider effective models that take into account quantum fluctuations of space and their physical consequences. This has been done, for instance, in [26] for the case of a fluctuating volume and its influence on the initial singularity.

According to Eqs. (3), the Green functions for the gravitational field are defined by means of the equations

$$\begin{gathered}\langle Q|T(\hat{G}_{\mu\nu})|Q\rangle=\varkappa\langle Q|T(\hat{T}_{\mu\nu})|Q\rangle,\\ \langle Q|T(\hat{g}(x_1)\cdot\hat{G}_{\mu\nu})|Q\rangle=\varkappa\langle Q|T(\hat{g}(x_1)\cdot\hat{T}_{\mu\nu})|Q\rangle,\\ \cdots=\cdots,\\ \langle Q|T(\text{a product of } g \text{ at different points}\end{gathered}$$

$$(x_1,\cdots,x_n)\cdot\hat{G}_{\mu\nu}\big)\big|Q\big\rangle$$
$$=\varkappa\big\langle Q\big|T\big(\text{a product of } g \text{ at different points}$$
$$(x_1,\cdots,x_n)\cdot\hat{T}_{\mu\nu}\big)\big|Q\big\rangle. \quad (5)$$

To treat any particular physical system, it is necessary to solve the above equations in which the quantum state $|Q\rangle$ will depend on the properties of the system under consideration. Moreover, all operators in (4) also depend on the properties of each particular physical system. In other words, the quantum information of a physical system is encoded in the set of operators $\hat{g}_{\mu\nu}$, $\hat{R}_{\mu\nu}\ldots$ and the quantum state $|Q\rangle$, which are in turn completely determined by the set of Green functions. In addition, analogous equations should be taken into account formatter fields, which will depend on the specific type of matter.

Notice that the Green functions are divergent in the perturbative approach due to the use of Feynman diagrams for their calculations. In the nonperturbative approach proposed here, it is not the case. For determination of the Green functions nonperturbatively, however, we must solve a set of highly complicated differential equations, implying probably new mathematical difficulties, which are absent in the perturbative case.

In a series of papers [24–27] it was shown that the set of equations (5) leads effectively to the appearance of modified theories of gravity. To exemplify the general procedure described in the previous section, we here present a brief review of the main steps leading to modified gravities.

In [25], it was shown that quantization of the Einstein-Hilbert Lagrangian

$$\mathcal{L}=-\frac{c^2}{2\varkappa}\sqrt{-g}R, \quad (6)$$

effectively leads to a modified Lagrangian of the form

$$\langle Q|\mathcal{L}_{\mathcal{G}}(g+\widehat{\delta g})|Q\rangle=\mathcal{L}_{\rm mod}$$
$$\approx-\frac{c^2}{2\varkappa}\sqrt{-g}\Big[R+\Big(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\Big)K^{\mu\nu}\Big], \quad (7)$$

where $K^{\mu\nu}$ is the quantum contribution determined by the equations

$$\langle Q|\widehat{\delta g}_{\mu\nu}|Q\rangle=K_{\mu\nu}, \quad (8)$$

$$\hat{g}_{\mu\nu}=\langle Q|g_{\mu\nu}|Q\rangle+\widehat{\delta g}_{\mu\nu}. \quad (9)$$

We here consider a stationary case, so that there is no need for the time-ordering operator.

An explicit expression for $K^{\mu\nu}$ can be obtained only in the framework of an exact theory of quantum gravity, which is not known. However, we can use the fact that (9) is a tensor equation, and therefore its both sides must have the same tensorial properties. Consequently, to construct the tensor $K_{\mu\nu}$ we have only two options, namely,

$$K_{\mu\nu}=F(R,R_{\mu\nu}R^{\mu\nu},\cdots)\,\langle Q\,|g_{\mu\nu}|\,Q\rangle\,, \quad (10)$$

$$K_{\mu\nu}=F(R,R_{\mu\nu}R^{\mu\nu},\cdots)\frac{R_{\mu\nu}}{R}, \quad (11)$$

where $F(R,R_{\mu\nu}R^{\mu\nu},\cdots)$ is an unknown function, and all tensors $R,R_{\mu\nu},\ldots$ are computed for the metric $\langle Q|g_{\mu\nu}|Q\rangle$.

Moreover, in [24] it was shown that the second-order deviations of the metric operator $\hat{g}_{\mu\nu}$

$$\hat{g}_{\mu\nu}\approx g_{\mu\nu}+\widehat{\delta g}_{\mu\nu}+\widehat{\delta^2 g}_{\mu\nu}, \quad (12)$$

correspond to the Lagrangian

$$\hat{\mathcal{L}}(g+\widehat{\delta g}+\widehat{\delta^2 g})\approx\mathcal{L}(g)+\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}\widehat{\delta g^{\mu\nu}}$$
$$+\frac{\delta^2\mathcal{L}}{\delta g^{\mu\nu}\delta g^{\rho\sigma}}\widehat{\delta g^{\mu\nu}}\widehat{\delta g^{\rho\sigma}}+\frac{\delta^2\mathcal{L}}{\delta^2 g^{\mu\nu}}\widehat{\delta^2 g^{\mu\nu}}. \quad (13)$$

The last term of this equation can be written as

$$\Big\langle Q\Big|\delta^2\hat{\mathcal{L}}(g)\Big|\,Q\Big\rangle=\frac{\delta^2\mathcal{L}(g)}{\delta g^{\mu\nu}\delta g^{\rho\sigma}}\Big\langle Q\Big|\widehat{\delta g^{\mu\nu}}\,\widehat{\delta g^{\rho\sigma}}\Big|\,Q\Big\rangle$$
$$+\frac{\delta^2\mathcal{L}}{\delta^2 g^{\mu\nu}}\Big\langle Q\Big|\widehat{\delta^2 g^{\mu\nu}}\Big|\,Q\Big\rangle$$
$$=-\frac{c^2}{2\varkappa}\sqrt{-g}\Big\{\Big[-\frac{1}{2}G_{\mu\nu}g_{\alpha\beta}\delta g^{\alpha\beta}+\delta R_{\mu\nu}$$
$$-\frac{R}{2}\delta g_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\delta R\Big]\delta g^{\mu\nu}+G_{\mu\nu}\delta^2 g^{\mu\nu}\Big\}. \quad (14)$$

Then, to calculate the explicit expression of the quantum contribution due to the last term of (12), we need to compute the two-point Green function

$$G_{\mu\nu,\rho\sigma}(x_1,x_2)=\big\langle Q\big|\widehat{\delta g_{\mu\nu}}(x_1)\cdot\widehat{\delta g_{\rho\sigma}}(x_2)\big|Q\big\rangle. \quad (15)$$

Considering that this is a tensor equation and assuming that in the first-order approximation the Green function can be expressed as a product of two functions, we obtain

$$G_{\mu\nu,\rho\sigma}\,(x_1,x_2)\approx P_{\mu\nu}(x_1)P_{\rho\sigma}(x_2). \quad (16)$$

Moreover, to find the Lagrangian for the modified gravity theory, we must also compute the quantum contribution

$$\Big\langle Q\Big|\widehat{\delta^2 g_{\mu\nu}}\Big|Q\Big\rangle=K_{\mu\nu}. \quad (17)$$

Then, considering the tensor character of Eqs. (16) and (17), we can in general assume that

$$P_{\mu\nu},K_{\mu\nu}$$
$$=F(R,R_{\mu\nu}R^{\mu\nu},\cdots)\begin{cases}\text{either } g_{\mu\nu},\\ \text{or } R_{\mu\nu}/R,\end{cases} \quad (18)$$

where $F(R, R_{\mu\nu}R^{\mu\nu}, \cdots)$ is an unknown function. Finally, we obtain the expression

$$\langle Q|\widehat{\delta^2\mathcal{L}}|Q\rangle = -\frac{c^2}{\varkappa}\Big\{ -\frac{3}{2}F^2\frac{R_{\alpha\beta}R^{\alpha\beta}}{R} + F\frac{R^{\alpha\beta}}{R}\Big[\frac{1}{2}\nabla_\alpha\nabla_\rho\Big(F\frac{R^\rho_\beta}{R}\Big) - \frac{1}{4}\nabla^\rho\nabla_\rho\Big(F\frac{R_{\alpha\beta}}{R}\Big) - \frac{1}{4}\nabla_\alpha\nabla_\beta F\Big] - \frac{F}{2}\nabla^\alpha\nabla^\beta\Big(F\frac{R_{\alpha\beta}}{R}\Big) + \frac{1}{2}R^\sigma_\alpha R_{\sigma\beta}R^{\alpha\beta}\frac{F^2}{R} - \frac{1}{2}R_{\alpha\rho\beta\sigma}R^{\rho\sigma}R^{\alpha\beta}\frac{F^2}{R^2} + \frac{F}{2}\nabla^\rho\nabla_\rho F + \frac{F^2R^2}{4}\Big\}. \quad (19)$$

This equation determines a modified theory of gravity in which the quantum contributions correspond to the average of the quantum fluctuations of the metric and of the two-point quantum correlations of the metric.

The Lagrangians (7) and (19) determine modified gravity theories with field equations obtained by means of variation with respect to $g_{\mu\nu}$. Any solution of these modified field equations describes a quantum gravitational system in which the physical properties of the quantum operator $\widehat{\delta g_{\mu\nu}}$ and of the quantum state $|Q\rangle$ are determined by Eqs. (8) and (18). Analogously, the properties of the operator $\widehat{\delta^2 g_{\mu\nu}}$ and the quantum state $|Q\rangle$ are determined by Eqs. (17) and (18).

2.2. Gauge Theories

In the case of gauge theories, nonperturbative quantum states defined in terms of Green functions can also be handled explicitly. For the sake of concreteness, let us consider an SU(3) gauge field. Then, the operator equation (2) leads to the Yang-Mills equations

$$D_\nu\widehat{F}^{A\mu\nu} = 0, \quad (20)$$

where

$$\hat{F}^B_{\mu\nu} = \partial_\mu\hat{A}^B_\nu - \partial_\nu\hat{A}^B_\mu + gf^{BCD}\hat{A}^C_\mu\hat{A}^D_\nu \quad (21)$$

is the field strength operator; $\hat{A}^B_\mu$ is the gauge potential operator; $B, C, D = 1, \ldots, 8$ are the SU(3) color indices; g is the coupling constant; and f^{BCD} are the structure constants for the SU(3) gauge group.

The Yang-Mills operator equations (20) are equivalent to an infinite set of equations for the Green functions, i.e.,

$$\begin{gathered}\big\langle T\big(D_\nu\widehat{F}^{A\mu\nu}(x)\big)\big\rangle = 0,\\ \big\langle T\big(\hat{A}^{B_1}_{\alpha_1}(x_1)D_\nu\widehat{F}^{A\mu\nu}(x)\big)\big\rangle = 0,\\ \big\langle T\big(\hat{A}^{B_1}_{\alpha_1}(x_1)\hat{A}^{B_2}_{\alpha_2}(x_2)D_\nu\widehat{F}^{A\mu\nu}(x)\big)\big\rangle = 0,\\ \ldots = 0,\\ \big\langle T\big(\hat{A}^{B_1}_{\alpha_1}(x_1)\ldots\hat{A}^{B_n}_{\alpha_n}(x_n)D_\nu\widehat{F}^{A\mu\nu}(x)\big)\big\rangle = 0,\\ \ldots = 0. \end{gathered} \quad (22)$$

This system possesses different particular solutions, each of which determines a particular quantum state. The physical properties of each state must be investigated separately; however, one can expect that some states will correspond to standard SU(3) configurations. Notice that the problem of divergences that occurs in the perturbative quantization approach is not present here because we use a different method to calculate the nonperturbative Green functions.

A construction similar to that described above was used in [24, 25] to demonstrate that modified gravity theories arise as a consequence of applying the nonperturbative quantization procedure to general relativity.

3. WHEELER-DEWITT EQUATIONS

One of the attempts to quantize gravity is the so-called Wheeler-DeWitt approach, which essentially consists in assuming the validity of the stationary Schrödinger equation of quantum mechanics in the framework of Einstein's theory of gravity. The starting point are the $\binom{0}{\mu}$ components of the Einstein equations in which the components of the spatial metric g_{ab} $(a, b, ... = 1, 2, 3)$ are considered as generalized configuration variables. Then, one introduces the operators

$$\hat{g}_{ab}(x) = g_{ab}(x), \quad \hat{p}^{ab}(x) = \frac{1}{im_P^2}\frac{\delta}{\delta g_{ab}(x)}. \quad (23)$$

The operator $\hat{p}^{ab}(x)$ is considered as being canonically conjugated to $\hat{g}_{ab}$ and is essentially given by the variational derivatives $\delta/\delta g_{ab}$. This operator replaces the derivatives $\dot{g}_{ab}$ in the components $\binom{0}{\mu}$ of Einstein's equations. As a result, one obtains the Wheeler-DeWitt equations, which are similar to the stationary Schrödinger equation and can be written as

$$\Big(-\frac{1}{2m_P^2}G_{ab,cd}\frac{\delta^2}{\delta g_{ab}\delta g_{cd}} - m_P^2\sqrt{g}^3R\Big)|\Psi[g_{ab}]\rangle = 0, \quad (24)$$

$$\Big(-\frac{2}{i}g_{ab}\nabla_c\frac{\delta}{\delta g_{bc}}\Big)|\Psi[g_{ab}]\rangle = 0, \quad (25)$$

where

$$G_{ab,cd} = g^{-1/2}(g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ab}g_{cd}) \quad (26)$$

is the local DeWitt supermetric, and ${}^3R$ is the 3D scalar curvature; $a, b, c, d = 1, 2, 3$ are spatial indices. The set of equations (25) represents the constraint equations.

We now compare the Wheeler-DeWitt approach with the nonperturbative quantization procedure described in Section 2. To this end, we write Eqs. (5) explicitly for $\mu = 0$, $\nu = 1, 2, 3$ as

$$\left\langle \hat{R}_{00} - \frac{1}{2}\hat{g}_{00}\hat{R} \right\rangle = 0, \tag{27}$$

$$\left\langle \hat{R}_{0a} - \frac{1}{2}\hat{g}_{0a}\hat{R} \right\rangle = 0. \tag{28}$$

This particular representation is to be compared with Eqs. (24) and (25). The most important difference between Eqs. (27), (28) and Eqs. (24), (25) is that the following operators are not equal:

$$\hat{g}_{ab}(x) \neq g_{ab}(x), \tag{29}$$

$$\hat{p}^{ab}(x) \neq \frac{1}{im_P^2}\frac{\delta}{\delta g_{ab}(x)}, \tag{30}$$

and for this reason they are completely different from the canonically conjugated operators (23).

Another important difference is that in the Wheeler-DeWitt approach one can define the operators $\hat{g}_{ab}$ and $\hat{g}^{ab}(x)$ independently of the quantum state $\Psi[g_{ab}]$. In the nonperturbative approach, it is not possible. Indeed, the result of solving Eqs. (5) is the entire set of Green functions that contain all quantum information on the corresponding physical system. It is not possible to write down separately the operators and the quantum states in a particular specific form. This implies that in the nonperturbative quantization approach á la Heisenberg, the Green functions are the only physical quantities that describe the physical system and its properties. *In particular, the definition of the quantum operators and their commutation relations are completely determined by the set of nonperturbative Green functions.*

4. ENERGY SPECTRUM AND PARTITION FUNCTION OF A QUANTUM MONOPOLE IN CONTACT WITH A THERMOSTAT

To demonstrate how one can calculate thermodynamic quantities for nonperturbatively quantized fields, we employ the simplest object supported by such fields, a quantum monopole [22].

4.1. Energy Spectrum of a Quantum Monopole

In general, it is not possible to solve the entire set of equations (22); however, one can consider the average quantum values as given by the classical value of the potential p lus a fluctuation term that can be used to truncate the set of equations. Indeed, this procedure has been performed in [23], leading to the result that Eqs. (22) reduce to a set of two equations, namely,

$$\tilde{D}_\nu F^{a\mu\nu} - \left[\left(m^2\right)^{ab\mu\nu} - \left(\mu^2\right)^{ab\mu\nu} \right] A^b_\nu = 0, \tag{31}$$

$$\Box\phi - \left(m^2_\phi\right)^{ab\mu\nu} A^a_\nu A^b_\mu \phi - \lambda\phi(M^2 - \phi^2) = 0, \tag{32}$$

where $\tilde{D}_\mu$ is the gauge derivative of the subgroup SU(2); $(m^2)^{ab\mu\nu}$, $(\mu^2)^{ab\mu\nu}$, and $(m^2_\phi)^{ab\mu\nu}$ are quantum corrections coming from the dispersions of the operators $\widehat{\delta A}^{a\mu}$ and $\hat{A}^{m\mu}$:

$$\hat{A}^{a\mu} = \left\langle \hat{A}^{a\mu} \right\rangle + i\widehat{\delta A}^{a\mu}, \tag{33}$$

$$\left\langle \hat{A}^{m\mu} \right\rangle = 0. \tag{34}$$

The quantum averaging $\langle \ldots \rangle$ in Eqs. (33) and (34) is understood as averaging over the nonperturbative quantum state defined in Section 2. Recall that such a quantum state is determined by the entire set of Green functions. In this case, the nonperturbative quantum state is approximately determined by the 2-point Green functions (35), (36) and the 4-point Green function (41) defined below.

The quantum corrections $(m^2)^{ab\mu\nu}$, $(\mu^2)^{ab\mu\nu}$, and $(m^2_\phi)^{ab\mu\nu}$ are determined by the 2-point Green functions

$$G^{mn\mu\nu}(y, x) = \left\langle \hat{A}^{m\mu}(y)\hat{A}^{n\nu}(x) \right\rangle, \tag{35}$$

$$G^{ab\mu\nu}(y, x) = \left\langle \widehat{\delta A}^{a\mu}(y)\widehat{\delta A}^{b\nu}(x) \right\rangle. \tag{36}$$

Since we are considering here only the stationary case, the time-ordering operator is absent. We approximate the above functions as follows:

$$G^{mn\mu\nu}(y, x) \approx -\Delta^{mn}\mathcal{A}^\mu\mathcal{A}^\nu\phi(y)\phi(x), \tag{37}$$

$$G^{ab\mu\nu}(y, x) \approx \Delta^{ab}\mathcal{B}^\mu\mathcal{B}^\nu, \tag{38}$$

where Δ^{ab} $(a, b = 2, 5, 7)$, Δ^{mn} $(m, n = 1, 3, 4, 6, 8)$ are constants; $\mathcal{A}_\mu\mathcal{A}^\nu, \mathcal{B}_\mu\mathcal{B}^\nu = \text{const}$.

The 2-point Green function (37) requires a more detailed discussion. The point is that the Green function $G^{mm\mu\mu}(x, x)$, being a dispersion, is defined in the following manner:

$$G^{mm\mu\mu}(x, x) = -\Delta^{mm}\left(\mathcal{A}^\mu\right)^2\phi(x)^2 < 0, \tag{39}$$

and it is a negative quantity. In quantum mechanics, the dispersion is defined by the known expression

$$\left\langle \left(\hat{L} - \langle L\rangle\right)^2 \right\rangle = \int \psi^* \left(\hat{L} - \langle L\rangle\right)^2 \psi dV > 0, \tag{40}$$

and it is a positive quantity if the operator $\hat{L}$ is a Hermitian conjugate operator. Similarly, in perturbative quantum field theory, the dispersion is also a

positive quantity. But this is not the case in our problem. The point is that, according to our approach, a nonperturbative quantum state including the dispersion is defined by solving the infinite set of equations (22). One may find that the corresponding solution has a negative dispersion. Such a quantum state may be called a *strange quantum state*. Notice that in our case, if the dispersion (39) is positive, the corresponding solutions obtained below do not exist. Note also that the quantum quantity $(m^2_\phi)^{ab\mu\nu}$ is associated with the Green function $G^{ab\mu\nu}$, and the negativity of the corresponding dispersion is analogous to the appearance of an imaginary mass for the corresponding quantum in perturbative quantum field theory. We notice that everything stated above applies equally well to a dispersion of the quantity $\hat{A}^a_\mu$ since we took into account the negativity of the corresponding dispersion by introducing the imaginary unit in the formula (33).

Notice that to derive Eq. (32), we use the following approximation of the 4-point Green function:

$$G^{mnpq}{}_{\mu\nu\rho\sigma}(x,y,z,u) = \left\langle \hat{A}^m_\mu(x)\hat{A}^n_\nu(y)\hat{A}^p_\rho(z)\hat{A}^q_\sigma(u)\right\rangle, \quad (41)$$

$$G^{(4)} \approx \frac{\lambda}{4}\big(G^{(2)} - M^2 big\big)^2, \quad (42)$$

where λ, M are some constants. A detailed procedure of obtaining Eqs. (31) and (32) can be found in [22, 23].

To obtain equations describing a quantum monopole, we employ the ansatz for the SU(2) gauge fields in the standard monopole form in spherical coordinates:

$$A^a_\mu = \frac{2}{g}[1-f(r)] \times \begin{pmatrix} 0 & 0 & 0 & -\sin^2\theta \\ 0 & 0 & \cos\varphi & -\sin\theta\cos\theta\sin\varphi \\ 0 & 0 & \sin\varphi & \sin\theta\cos\theta\cos\varphi \end{pmatrix}, \quad (43)$$

$$\phi = \psi(r)/g, \quad (44)$$

where g is the coupling constant. Here $a = 2, 5, 7$, and the spacetime index $\mu = t, r, \theta, \varphi$. Also, we use the following approximation for Δ^{AB} $(A, B = 1, 2, \ldots, 8)$:

$$\Delta^{AB} = \mathrm{diag}(\Delta_{11}, \delta_2, \Delta_{33}, \Delta_{44}, \delta_5, \Delta_{66}, \delta_7, \Delta_{88}) \quad (45)$$

with $\Delta_{66} = \Delta_{44}$, $\Delta_{44} + \Delta_{88} = \Delta_{11} + \Delta_{33}$, and the vectors $\mathcal{A}_\mu, \mathcal{B}_\mu$

$$\mathcal{A}_\mu = (\mathcal{A}_0, \mathcal{A}_1, 0, 0), \quad (46)$$

$$\mathcal{B}_\mu = (\mathcal{B}_0, 0, 0, 0). \quad (47)$$

As a result, we obtain the following set of equations describing the quantum monopole:

$$-f'' + \frac{f(f^2-1)}{x^2} - m^2(1-f)\tilde{\phi}^2 = -\tilde{\mu}^2(1-f), \quad (48)$$

$$\tilde{\phi}'' + \frac{2}{x}\tilde{\phi}' = \tilde{\phi}\left[m^2\frac{(1-f)^2}{x^2} + \tilde{\lambda}(\tilde{\phi}^2 - \tilde{M}^2)\right], \quad (49)$$

where we have introduced the dimensionless variables $x = r/r_0$, $\tilde{\phi} = r_0\psi$, $\tilde{\mu} = r_0\mu$, $\tilde{M} = gr_0M$, $\tilde{\lambda} = \lambda/g^2$. Here r_0 is a characteristic size of the system. It can be rewritten in terms of some field constant ψ_1 as $r_0 = \psi_1^{-1}$, which will be used below in defining the partition function of the monopole. Note also that, for simplicity, hereafter we set $m_\phi = m$.

The obtained set of equations is solved numerically as a nonlinear eigenvalue problem for the eigenvalues $\tilde{\mu}$ and $\tilde{M}$ and the eigenfunctions $f(x)$ and $\tilde{\phi}(x)$. To do that, we choose the following boundary conditions at the center::

$$f(0) = 1, \quad f'(0) = 0, \quad \tilde{\phi}(0) \equiv \tilde{\phi}_0 = \mathrm{const}, \quad \tilde{\phi}'(0) = 0. \quad (50)$$

For numerical solving, it is necessary to have a solution in the neighborhood of the center, which can be presented as a power series,

$$f(x) = 1 + f_2\frac{r^2}{2} + \ldots = 1 + \tilde{f}_2\frac{x^2}{2} + \ldots, \quad (51)$$

$$\tilde{\phi}(x) = \tilde{\phi}_0 + \tilde{\phi}_2\frac{x^2}{2} + \ldots, \quad (52)$$

where the expansion coefficient f_2 is arbitrary (in dimensionless units, $\tilde{f}_2 = f_2/\psi_1^2$) and $\tilde{\phi}_2 = \tilde{\lambda}\tilde{\phi}_0(\tilde{\phi}_0^2 - \tilde{M}^2)/3$. Using these boundary conditions, we have obtained families of solutions of Eqs. (48) and (49) for different $\tilde{f}_2$, shown in Fig. 1.

The asymptotic behavior of the eigenfunctions $f(x), \tilde{\phi}(x)$ can be found by analyzing the set of equations (48) and (49) in the following form:

$$f(x) \approx 1 - f_\infty e^{-x\sqrt{m^2\tilde{M}^2 - \tilde{\mu}^2}}, \quad (53)$$

$$\tilde{\phi}(x) \approx \tilde{M} - \tilde{\phi}_\infty\frac{e^{-x\sqrt{2\tilde{\lambda}\tilde{M}^2}}}{x}. \quad (54)$$

Here $f_\infty, \tilde{\phi}_\infty$ are constants whose values are determined by the parameter $\tilde{f}_2$.

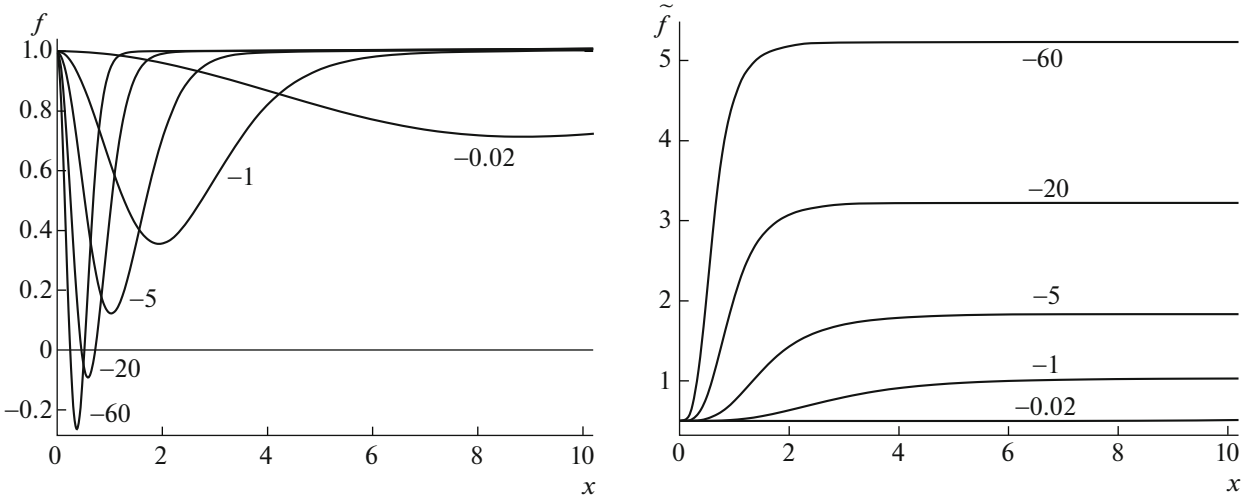

Fig. 1. The eigenfunctions $f(x)$ and $\tilde{\phi}(x)$ for different values of $\tilde{f}_2 = -0.02, -1.0, -5.0, -20.0, -60.0$ and $\tilde{\phi}_0 = 0.5$, $m = 2$.

The dimensional energy density of the quantum monopole under study is given by the expression

$$L_{\rm mp} = \epsilon_{\rm mp} = \frac{1}{g^2}\left[\frac{1}{2}\frac{f'^2}{r^2} + \frac{\psi'^2}{2} + \frac{1}{4}\frac{\left(f^2-1\right)^2}{r^4} + \frac{m^2}{2}\frac{(f-1)^2}{r^2}\psi^2 - \frac{1}{2}\frac{\mu^2}{r^2}(f-1)^2 + \frac{\tilde{\lambda}}{4}\left(\psi^2 - g^2M^2\right)^2 + g^2\epsilon_\infty\right], \qquad (55)$$

and it is plotted in Fig. 2 in a dimensionless form. Using (55), the total energy is calculated as follows:

$$E_{\rm mp} = 4\pi\int\limits_0^\infty \epsilon_{\rm mp} r^2 dr = \frac{4\pi}{g^2 r_0}\int\limits_0^\infty \tilde{\epsilon}_{\rm mp} x^2 dx = \frac{4\pi\psi_1}{g^2}\int\limits_0^\infty \tilde{\epsilon}_{\rm mp} x^2 dx = \frac{4\pi\hbar c\psi_1}{\tilde{g}^2}\int\limits_0^\infty \tilde{\epsilon}_{\rm mp} x^2 dx = \frac{4\pi\hbar c\psi_1}{\tilde{g}^2}\tilde{E}_{\rm mp}, \qquad (56)$$

where we have introduced the dimensionless coupling constant $\tilde{g}^2 = \hbar c g^2$; $\tilde{\epsilon}_{\rm mp}$ and $\tilde{E}_{\rm mp}$ are the dimen-

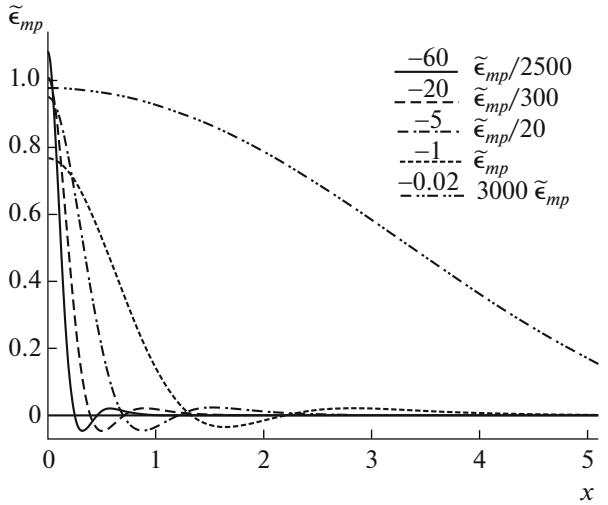

Fig. 2. The dimensionless monopole energy density $\tilde{\epsilon}_{\rm mp}(x)$ from (55) for different values of $\tilde{f}_2 = -0.02, -1.0, -5.0, -20.0, -60.0$.

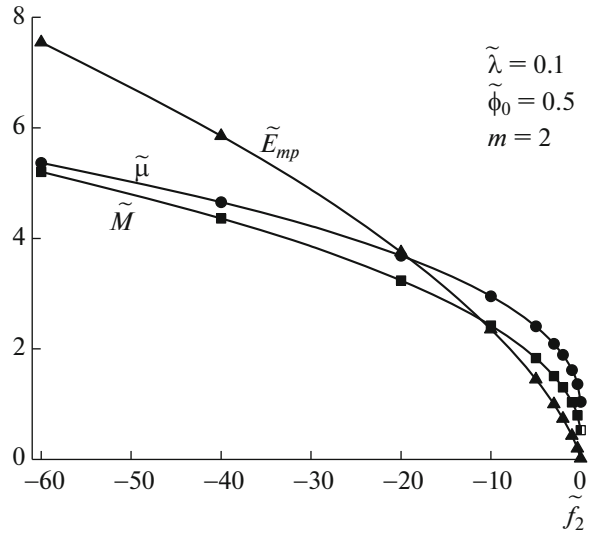

Fig. 3. Dependences of the monopole total energy $\tilde{E}_{\rm mp}$ from (56) and of the eigenvalues $\tilde{\mu}$, $\tilde{M}$ on the parameter $\tilde{f}_2$.

Table 1. The eigenvalues $\tilde{\mu}$, $\tilde{M}$ and the energy $\tilde{E}_{\rm mp}$ as functions of the parameter $\tilde{f}_2$ for the monopole

$\tilde{f}_2$	-60	-40	-20	-10	-5
$\tilde{M}$	5.202	4.36	3.238	2.421	1.833
$\tilde{\mu}$	5.364585475	4.65430559	3.68434345	2.95400186	2.406699273
$\tilde{E}_{mp}$	7.5457	5.85209	3.75006	2.35408	1.450611
$\tilde{f}_2$	-3	-2	-1.6	-1.4	-1.2
$\tilde{M}$	1.508	1.3029	1.2051	1.152	1.095
$\tilde{\mu}$	2.09391213	1.8908366	1.791790814	1.737779882	1.6793213
$\tilde{E}_{mp}$	1.00007	0.738754	0.622474	0.560275	0.496008
$\tilde{f}_2$	-1	-0.9	-0.8	-0.6	-0.4
$\tilde{M}$	1.0334	1	0.965	0.8885	0.7982
$\tilde{\mu}$	1.61522	1.5805559	1.5437766	1.4619973	1.364495
$\tilde{E}_{mp}$	0.42951	0.394122	0.35823	0.283069	0.200936
$\tilde{f}_2$	-0.2	-0.1	-0.05	-0.03	-0.02
$\tilde{M}$	0.687	0.6123	0.565	0.54228	0.52945
$\tilde{\mu}$	1.237047	1.1492464	1.09011125	1.0601184	1.042631
$\tilde{E}_{mp}$	0.11228	0.0607634	0.0332751	0.0216438	0.0199801

sionless energy density and total energy, respectively. Figure 3 shows the dependences of the total energy and of the eigenvalues $\tilde{\mu}, \tilde{M}$ on the parameter $\tilde{f}_2$, from which one can see that the limiting values are

$$\tilde{\mu} \xrightarrow{\tilde{f}_2\to 0} 1, \quad \tilde{M} \xrightarrow{\tilde{f}_2\to 0} \tilde{\phi}_0, \quad \tilde{E}_{\rm mp} \xrightarrow{\tilde{f}_2\to 0} 0. \quad (57)$$

Table 1 shows the eigenvalues $\tilde{\mu}, \tilde{M}$ and the energy $\tilde{E}_{\rm mp}$ as functions of the parameter $\tilde{f}_2$.

4.2. Energy Spectrum of a Flux Tube

In this section we consider an infinite flux tube filled with a transverse magnetic field. Such a flux tube of finite length can appear between a monopole and an antimonopole in the vacuum of QCD.

To obtain the corresponding solutions, we use the same two-equation approximation (31) and (32) and the following ansatz describing the transverse magnetic field creating an infinite flux tube:

$$A_\varphi^2 = \frac{\rho}{g} w(\rho), \quad (58)$$

$$\phi = \frac{\psi(\rho)}{g} \quad (59)$$

in cylindrical coordinates z, ρ, φ. The matrix Δ^{AB} is chosen in the form (45) with arbitrary Δ_{11}, Δ_{33}, Δ_{44}, Δ_{66}, Δ_{88} and the vectors $\mathcal{A}_\mu, \mathcal{B}_\mu$

$$\mathcal{A}_\mu = (\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, 0), \quad (60)$$

$$\mathcal{B}_\mu = (\mathcal{B}_0, \mathcal{B}_1, 0, 0). \quad (61)$$

Using the ansatz (58)–(59) and the auxiliary quantities (60)–(61), we obtain the following equations describing the infinite flux tube filled with a transverse magnetic field and embedded in a quantum condensate described by the scalar field ϕ:

$$-\tilde{w}'' - \frac{\tilde{w}'}{x} + \frac{\tilde{w}}{x^2} + m^2\tilde{\phi}^2\tilde{w} = \tilde{\mu}^2\tilde{w}, \quad (62)$$

$$\tilde{\phi}'' + \frac{\tilde{\phi}'}{x} = \tilde{\phi}\left[m^2\tilde{w}^2 + \tilde{\lambda}\left(\tilde{\phi}^2 - \tilde{M}^2\right)\right]. \quad (63)$$

In these equations we have used the dimensionless variables $x = \rho/\rho_0$, $\tilde{w} = \rho_0 w$, $\tilde{\phi} = \rho_0\psi$, $\tilde{\mu} = \rho_0\mu$, $\tilde{M} = g\rho_0 M$, $\tilde{\lambda} = \lambda/g^2$.

Just as in the previous section, the set of equations (62) and (63) is solved as a nonlinear eigenvalue problem with the boundary conditions

$$\tilde{w}(0) = 0, \quad \tilde{w}'(0) = w_1,$$
$$\tilde{\phi}(0) \equiv \tilde{\phi}_0 = \text{const}, \quad \tilde{\phi}'(0) = 0. \quad (64)$$

To obtain numerical solutions, we use a Taylor expansion of the functions $\tilde{w}(x)$, $\tilde{\phi}(x)$ at the origin:

$$\tilde{w}(x) = \tilde{w}_1 x + \tilde{w}_3\frac{x^3}{6} + \ldots, \quad (65)$$

$$\tilde{\phi}(x) = \tilde{\phi}_0 + \tilde{\phi}_2\frac{x^2}{2} + \ldots, \quad (66)$$

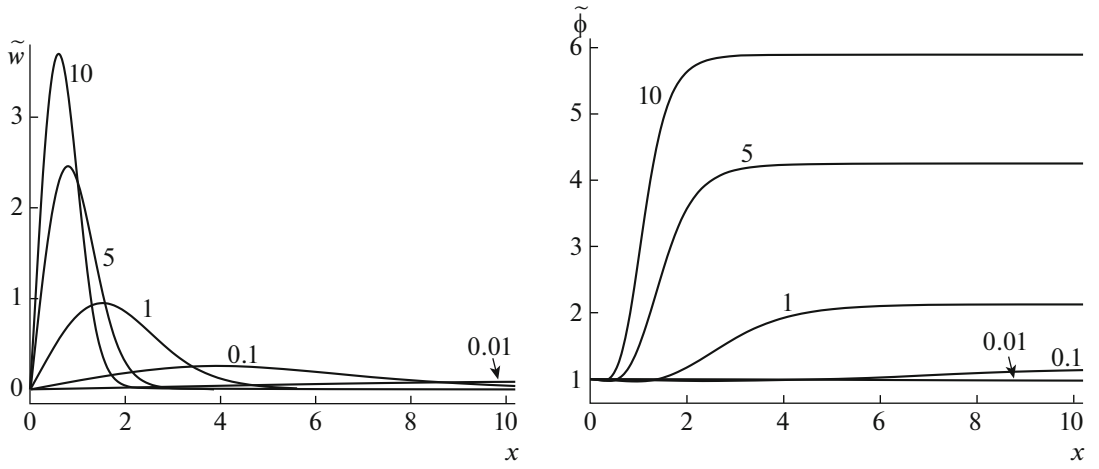

Fig. 4. The eigenfunctions $\tilde{w}(x)$ and $\tilde{\phi}(x)$ for different values of $\tilde{w}_1 = 0.01, 0.1, 1.0, 5.0, 10.0$ and $\tilde{\phi}_0 = 1$, $m = 1$.

where $\tilde{w}_1$ is arbitrary and $\tilde{\phi}_2 = \tilde{\lambda}\tilde{\phi}_0(\tilde{\phi}_0^2 - \tilde{M}^2)/2$.The corresponding numerical solutions are given in Fig. 4.

The asymptotic behavior of the eigenfunctions $\tilde{w}(x), \tilde{\phi}(x)$ can be found from the analysis of Eqs. (62) and (63) in the following form:

$$\tilde{w}(x) \approx \tilde{w}_\infty \frac{e^{-x\sqrt{m^2\tilde{M}^2 - \tilde{\mu}^2}}}{\sqrt{x}}, \tag{67}$$

$$\tilde{\phi}(x) \approx \tilde{M} - \tilde{\phi}_\infty \frac{e^{-x\sqrt{2\tilde{\lambda}\tilde{M}^2}}}{\sqrt{x}}. \tag{68}$$

Here $\tilde{w}_\infty, \tilde{\phi}_\infty$ are constants whose values are determined by the parameter $\tilde{w}_1$.

With the above solutions at hand, one can calculate the distribution of the transverse color magnetic field H_z^2, defined as

$$H_z^2 = \frac{1}{g}\left(w' + \frac{w}{\rho}\right). \tag{69}$$

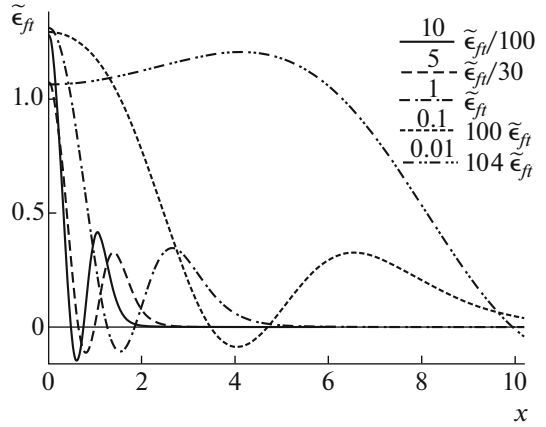

Fig. 5. The dimensionless flux tube energy density $\tilde{\epsilon}_{\text{ft}}(x)$ for different values of $\tilde{w}_1 = 0.01, 0.1, 1.0, 5.0, 10.0$.

The dimensional flux tube energy density is given by the expression

$$L_{\text{ft}} = \epsilon_{\text{ft}} = \frac{1}{g^2}\left[\frac{w'^2}{2} + \frac{\psi'^2}{2} + \frac{w^2}{2r^2} + \frac{m^2}{2}w^2\psi^2 - \frac{\mu^2}{2}w^2 + \frac{\tilde{\lambda}}{4}\left(\psi^2 - g^2M^2\right)^2 + g^2\epsilon_\infty\right], \tag{70}$$

and it is shown in Fig. 5 in a dimensionless form. Using (70), the linear energy density is calculated as follows:

$$E_{\text{ft}} = 2\pi\int_0^\infty \epsilon_{\text{ft}} r dr = \frac{2\pi}{g^2\rho_0^2}\int_0^\infty \tilde{\epsilon}_{\text{ft}} x dx = \frac{2\pi\hbar c}{\tilde{g}^2\rho_0^2}\int_0^\infty \tilde{\epsilon}_{\text{ft}} x dx = \frac{2\pi\hbar c}{\tilde{g}^2\rho_0^2}\tilde{E}_{\text{ft}}, \tag{71}$$

where $\tilde{\epsilon}_{\text{ft}}$ and $\tilde{E}_{\text{ft}}$ are the dimensionless energy density and linear energy density, respectively. Fig. 6 shows the dependences of the linear energy density and of the eigenvalues $\tilde{\mu}, \tilde{M}$ on the parameter $\tilde{w}_1$, from which one can observe the following limiting values:

$$\tilde{\mu} \xrightarrow{\tilde{w}_1 \to 0} 1, \quad \tilde{M} \xrightarrow{\tilde{w}_1 \to 0} \tilde{\phi}_0, \quad \tilde{E}_{\text{ft}} \xrightarrow{\tilde{w}_1 \to 0} 0. \tag{72}$$

Table 2 shows the eigenvalues $\tilde{\mu}, \tilde{M}$ and the energy $\tilde{E}_{\text{ft}}$ as functions of the parameter $\tilde{w}_1$.

4.3. Partition Function of the Quantum Monopole

To calculate the partition function of the quantum monopole, it is necessary to know on what parameters its energy depends. Then the partition function will be given by an integral over this parameter, since in the presence of statistical fluctuations the energy of the

Table 2. The eigenvalues $\tilde{\mu}$, $\tilde{M}$ and the linear energy density $\tilde{E}_{\rm ft}$ as functions of the parameter $\tilde{w}_1$ for the flux tube

$\tilde{w}_1$	0.01	0.02	0.04	0.06	0.08	0.1	0.2
$\tilde{M}$	1.008006233	1.0229746	1.0571195	1.092076	1.126168	1.159194	1.31045
$\tilde{\mu}$	0.996365	1.001275	1.017275	1.03485	1.052685	1.070163	1.14936
$\tilde{E}_{ft}$	0.0099494	0.0198091	0.0399769	0.0618725	0.0848041	0.108471	0.242524
$\tilde{w}_1$	0.5	1	2	5	10	20	40
$\tilde{M}$	1.67313	2.1319048	2.8212	4.24578	5.8864985	8.2323	11.5705
$\tilde{\mu}$	1.3346237	1.5621423	1.8948609	2.559342415	3.2980544	4.31989	5.726869
$\tilde{E}_{ft}$	0.737779	1.73539	4.07002	12.3594	28.2219	63.5609	141.505

system, and hence the corresponding parameters, are varying.

According to Eq. (56), the total energy of the quantum monopole consists of finite and infinite parts:

$$E_{\rm mp} = \frac{4\pi\hbar c\psi_1}{\tilde{g}^2}\left(\int\limits_0^\infty x^2\tilde{\epsilon}dx + \int\limits_0^\infty x^2\tilde{\epsilon}_\infty dx\right) = E_1 + E_2. \tag{73}$$

Here the first term, E_1, is finite, and the second one, E_2, is infinite. The total energy $E_{\rm mp}$ depends on the parameters f_2 and ψ_1. When statistical fluctuations occur, the energy fluctuation $\delta E_{\rm mp}$ should be finite, leading to changes in the parameters f_2 and ψ_1. But as ψ_1 varies, the second integral in (73) changes by an infinite amount. It is clear that when statistical fluctuations are present, the thermostat cannot provide an infinite energy, and therefore the parameter ψ_1 cannot vary and must remain constant under statistical fluctuations.

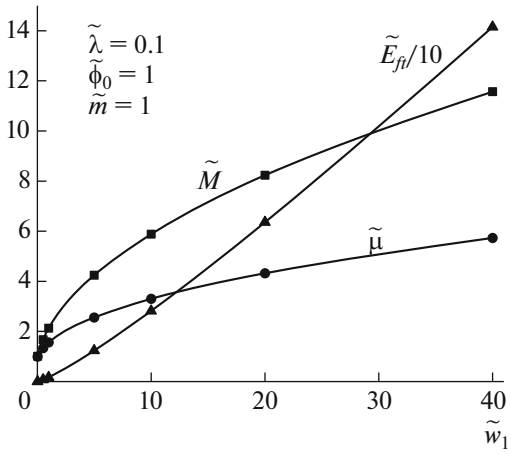

Fig. 6. Dependences of the eigenvalues $\tilde{\mu}, \tilde{M}$ and of the linear energy density $\tilde{E}_{\rm ft}$ on the parameter $\tilde{w}_1$.

In general, the partition function is calculated as

$$Z = \int e^{-\frac{H(\pi_A,\phi^A)}{kT}}\prod d\pi_A d\phi^A, \tag{74}$$

where $H(\pi_A, \phi^A)$ is the Hamiltonian of the system, ϕ^A, π_A are some generalized coordinates and momenta determining the energy of the system, and A is a collective index for all available indices. Consistent with the above, in our case the energy of the quantum monopole is determined by the parameter f_2. Hence, in the first approximation, the partition function can be calculated in the following way:

$$Z_{\rm mp}(T) \approx \frac{1}{\psi_1^2}\int e^{-\frac{E_1(f_2)}{kT}}df_2 = \int e^{-\frac{\tilde{E}_1(\tilde{f}_2)}{\tilde{T}}}d\tilde{f}_2 = \tilde{Z}_{\rm mp}(\tilde{T}). \tag{75}$$

Here $\tilde{E}_1$ is the dimensionless energy E_1 from (73), and $\tilde{T} = kT\tilde{g}^2/(4\pi\hbar c\psi_1)$ is the dimensionless temperature.

Having numerical values of $\tilde{E}_1$ in hand, we can evaluate the integral (75) only for the values of the temperature satisfying the conditions

$$\frac{\tilde{E}_{1,max}}{\tilde{T}} \gg 1, \quad T \ll \frac{4\pi\hbar c\psi_1}{k\tilde{g}^2}\tilde{E}. \tag{76}$$

Here $\tilde{E}_{1,\rm max}$ is the maximum value of the dimensionless energy (56) obtained by numerically solving Eqs. (48) and (49). For example, for $\psi_1 \approx 10^{15}$ m^{-1}, which corresponds to the typical length-scale of QCD $r_0 \approx 10^{-15}$ m, we have

$$T_{\rm max} \approx \frac{4\pi\hbar c\psi_1}{k\tilde{g}^2}\tilde{E} \approx 10^{15}{\rm K}. \tag{77}$$

The internal energy is calculated as follows:

$$U_{\rm mp}(T) = \frac{1}{Z_{\rm mp}}\int E_1(f_2)e^{-\frac{E_1(f_2)}{kT}}d\tilde{f}_2$$

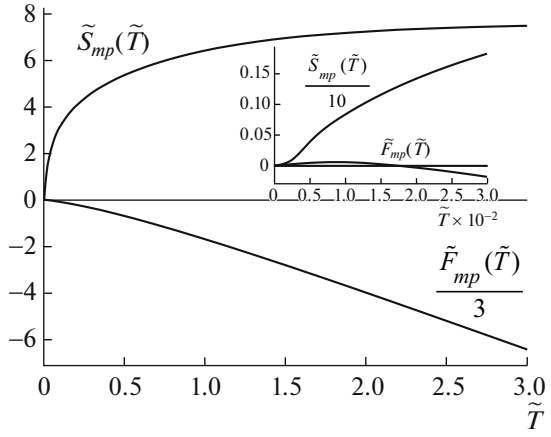

Fig. 7. The dimensionless internal energy $\tilde{U}_{\rm mp}(\tilde{T})$ and the partition function $\tilde{Z}_{\rm mp}(\tilde{T})$ of a single monopole.

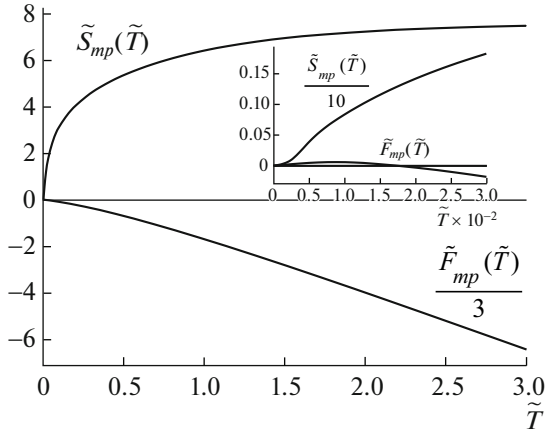

Fig. 8. The dimensionless entropy $\tilde{S}_{\rm mp}(\tilde{T})$ and the Helmholtz free energy $\tilde{F}_{\rm mp}(\tilde{T})$ of a single monopole. The inset shows the behavior of the functions in the neighbourhood of the center.

$$= \frac{\int E_1(f_2) e^{-\frac{E_1(f_2)}{kT}} d\tilde{f}_2}{\int e^{-\frac{E_1(f_2)}{kT}} d\tilde{f}_2} = \frac{4\pi\hbar c\psi_1}{\tilde{g}^2}$$

$$\times \frac{\int \tilde{E}_1(\tilde{f}_2) e^{-\frac{\tilde{E}_1(\tilde{f}_2)}{\tilde{T}}} d\tilde{f}_2}{\int e^{-\frac{\tilde{E}_1(\tilde{f}_2)}{\tilde{T}}} d\tilde{f}_2} = \frac{4\pi\hbar c\psi_1}{\tilde{g}^2} \tilde{U}_{\rm mp}(\tilde{T}). \quad (78)$$

The dependences of the dimensionless partition function $\tilde{Z}_{\rm mp}$ from (75) and of the dimensionless internal energy $\tilde{U}_{\rm mp}$ from (78) on the temperature $\tilde{T}$ are shown in Fig. 7.

The entropy $S_{\rm mp}(T)$ and the Helmholtz free energy $F_{\rm mp}(T) = U_{\rm mp}(T) - TS_{\rm mp}(T)$ are calculated in the following way:

$$S_{\rm mp}(T) = k\left(\ln Z_{\rm mp}(T) + \frac{U_{\rm mp}(T)}{kT} \right)$$

$$= k\left(\ln \tilde{Z}_{\rm mp}(\tilde{T}) + \frac{\tilde{U}_{\rm mp}(\tilde{T})}{\tilde{T}} \right) = k\tilde{S}_{\rm mp}(\tilde{T}), \quad (79)$$

$$F_{\rm mp}(T) = -kT \ln Z_{\rm mp}(T)$$

$$= -\frac{4\pi\hbar c\psi_1}{\tilde{g}^2} \tilde{T} \ln \tilde{Z}_{\rm mp}(\tilde{T}) = \frac{4\pi\hbar c\psi_1}{\tilde{g}^2} \tilde{F}_{\rm mp}(\tilde{T}). \quad (80)$$

The results of calculations indicate that as $\tilde{T} \to 0$ the entropy behaves in a regular manner,

$$\tilde{S}_{\rm mp}(\tilde{T}) \xrightarrow{\tilde{T}\to 0} \tilde{S}_1 = \text{const}.$$

This constant can be set to zero by the corresponding redefinition of the partition function from (75): $Z_{\rm mp} \to e^{\tilde{S}_1} Z_{\rm mp}$. The resulting free energy and entropy are shown in Fig. 8 for the choice $\tilde{S}_1 \approx 3.6$.

In this connection it is interesting to note that, unlike a classical system where the entropy demonstrates a singular behavior as the temperature goes to zero, for a nonperturbatively quantized system considered here the entropy is regular as $T \to 0$.

Notice also that since the Helmholtz free energy does not depend on the volume, the pressure p for the system under consideration is equal to zero:

$$p = -\left(\frac{\partial F(T)}{\partial V} \right)_T = 0. \quad (81)$$

Using this fact, we can verify our numerical calculations with the second law of thermodynamics

$$T\frac{dS}{dT} = \frac{dU}{dT}. \quad (82)$$

To calculate the left- and right-hand sides of this equation, we have employed numerically computed values of $S(T)$ and $U(T)$. The results of calculations indicate a perfect agreement of the left- and right-hand sides of (82). This also confirms that the pressure of the system consisting of one monopole is zero, according to Eq. (81).

5. PARTITION FUNCTION OF A NONRELATIVISTIC DILUTE GAS OF NONINTERACTING MONOPOLES

In this section we consider a nonrelativistic dilute gas of noninteracting monopoles enclosed in a volume V. The total energy of such a physical system consists of the energy of n monopoles [each having the energy determined by Eq. (56)], of the kinetic energy of the monopoles (calculated here in the approximation of pointlike particles), and of the energy of a quantum condensate with the energy density ϵ_∞:

$$E_{\rm gas}\left(\vec{p}_i, (f_2)_i\right) = \sum_{i=1}^{n} \left[E_{\rm mp}\Big((f_2)_i\Big) + \frac{p_i^2}{2(m_{\rm mp})_i (f_2)_i} \right]$$

$$+\epsilon_\infty V = E_1\Big(\vec{p}_i,(f_2)_i\Big)+E_2. \tag{83}$$

Here $(m_{\rm mp})_i$ is the mass of i-th monopole, $E_1(\vec{p}_i,(f_2)_i)$ is the energy of all n monopoles, and $E_2=\epsilon_\infty V$ is the energy of the condensate. To simplify the calculation of its kinetic energy, we assume that the monopole is a pointlike particle. The monopole mass is related to its energy by the expression $m_{\rm mp}=E_{\rm mp}/c^2$. Then the partition function is given by the following integral:

$$Z_{\rm gas}=\frac{1}{n!}\frac{1}{(2\pi\hbar)^{3n}}\int e^{-\frac{E_{\rm gas}\left(\vec{p}_i,(f_2)_i\right)}{kT}} \times\prod_{i=1}^{n}dp_x^i dp_y^i dp_z^i dx^i dy^i dz^i d(\tilde{f}_2)_i$$
$$=\frac{e^{-\frac{\epsilon_\infty V}{kT}}}{n!}\left[\frac{V}{(2\pi\hbar)^3}\int e^{-\frac{E_1(p,f_2)}{kT}}d^3p\,d\tilde{f}_2\right]^n. \tag{84}$$

In Section 4.3 we have calculated the partition function for one quantum monopole assuming that the energy of the quantum condensate, into which the quantum monopole is embedded, does not fluctuate. Continuing working within this assumption, here we calculate the partition function for n quantum monopoles embedded in the condensate filling the volume V. This means that we *calculate the partition function against the background of the quantum condensate.*

Integration over the momentum p in (84) is performed in the standard way and yields

$$Z_{\rm gas}=\frac{e^{-\frac{\epsilon_\infty V}{kT}}}{n!}\left[\frac{V}{\hbar^3}\left(\frac{kT}{2\pi c^2}\right)^{3/2} \times\int E_{\rm mp}^{3/2}(f_2)e^{-\frac{E_{\rm mp}(f_2)}{kT}}d\tilde{f}_2\right]^n$$
$$=\frac{e^{-\frac{\epsilon_\infty V}{kT}}}{n!}\left[\frac{V}{\hbar^3}\left(\frac{kT}{2\pi c^2}\right)^{3/2}\overline{E_{\rm mp}^{3/2}}(T)\right]^n$$
$$=\frac{e^{-\gamma}}{n!}\left[\frac{\gamma\,(kT)^{5/2}}{(2\pi)^{3/2}\hbar^3\epsilon_\infty c^3}\overline{E_{\rm mp}^{3/2}}\Big(T\Big)\right]^n. \tag{85}$$

Here $\gamma=\dfrac{\epsilon_\infty V}{kT}$ and

$$\overline{E_{\rm mp}^{3/2}}\Big(T\Big)=\int E_{\rm mp}^{3/2}(f_2)e^{-\frac{E_{\rm mp}(f_2)}{kT}}d\tilde{f}_2$$
$$=\left(\frac{4\pi\hbar c\psi_1}{\tilde{g}^2}\right)^{3/2}\int\tilde{E}_{\rm mp}^{3/2}e^{-\frac{\tilde{E}_{\rm mp}}{\tilde{T}}}d\tilde{f}_2$$
$$=\left(\frac{4\pi\hbar c\psi_1}{\tilde{g}^2}\right)^{3/2}\widetilde{E_{\rm mp}^{3/2}}\Big(\tilde{T}\Big) \tag{86}$$

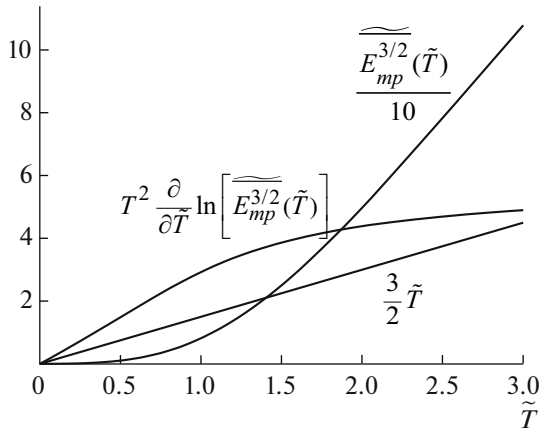

Fig. 9. Dependences of $\tilde{T}^2\dfrac{\partial}{\partial\tilde{T}}\ln\left[\widetilde{E_{\rm mp}^{3/2}}(\tilde{T})\right]$, $(3/2)\tilde{T}$, and $\widetilde{E_{\rm mp}^{3/2}}(\tilde{T})$ on the temperature for the monopole gas.

is the nonnormalized average statistical value of the monopole energy $E_{\rm mp}$ which depends on the temperature (here the term "nonnormalized" means that in defining the average statistical value we do not perform a division by $Z_{\rm gas}$). Fig. 9 shows the dependence of $\widetilde{E_{\rm mp}^{3/2}}$ on the dimensionless temperature $\tilde{T}$. It is seen that the appearance of the factor $\overline{E_{\rm mp}^{3/2}}(T)$, which depends on the temperature, results in a considerable difference between the partition functions of the monopole gas and of a classical monoatomic perfect gas for which $Z\sim T^{3n/2}$. Physically this difference is associated with the fact that the monopole is not a pointlike particle, and the energy gained from or returned to the thermostat changes not only the kinetic energy of monopoles but their internal energy as well.

Let us now calculate the pressure $p_{\rm gas}$ for the dilute gas of monopoles under consideration,

$$p_{\rm gas}=-\left(\frac{\partial F_{\rm gas}(T)}{\partial V}\right)_T, \tag{87}$$

where

$$F_{\rm gas}(T)=-kT\ln Z_{\rm gas}(T). \tag{88}$$

Substituting the expression for $Z_{\rm gas}$ (85) into (88), we have from (87)

$$p_{\rm gas}=-\epsilon_\infty+\frac{nkT}{V}. \tag{89}$$

Comparing this with the expression for a classical monoatomic ideal gas, one can observe the emergence of the term $-\epsilon_\infty$ associated with the energy of the quantum condensate.

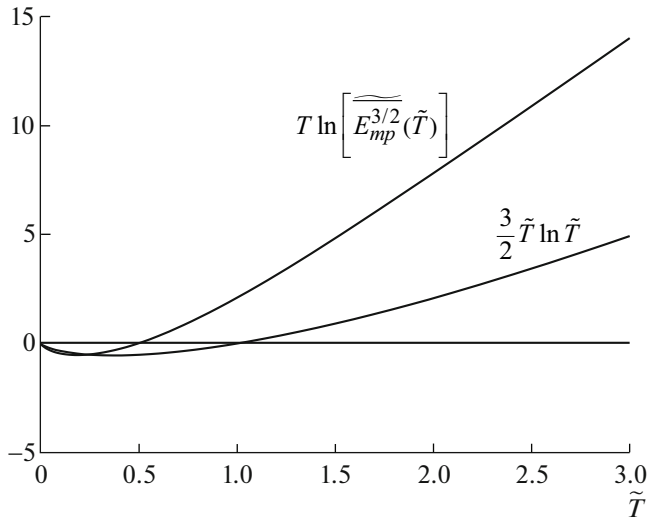

Fig. 10. Dependences of $\tilde{T}\ln\left[\widetilde{E^{3/2}_{\rm mp}}(\tilde{T})\right]$ and $(3/2)\tilde{T}\ln\tilde{T}$ on the temperature for the monopole gas.

In turn, the expression for the Helmholtz free energy (88) will now be

$$F_{\rm gas}(T)=\epsilon_\infty V-nkT\Bigg\{\ln\left[\overline{E^{3/2}_{\rm mp}}(T)\right]+\ln\frac{V}{V_0}+\frac{3}{2}\ln\frac{T}{T_0}\Bigg\}. \tag{90}$$

Here the first and second terms are contributions coming from the condensate ϕ and from a single monopole, respectively; the last term describes the dilute gas of noninteracting monopoles; V_0, T_0 depend on the constants entering into $Z_{\rm gas}$. The profile of the dependence of $\tilde{T}\ln\left[\widetilde{E^{3/2}_{\rm mp}}(\tilde{T})\right]$ on the dimensionless temperature $\tilde{T}$ is shown in Fig. 10, where the behavior of the fourth term from (90) is also given for comparison.

Finally, the internal energy is calculated as follows:

$$U_{\rm gas}(T)=-\frac{\partial\ln Z_{\rm gas}}{\partial\beta}=\epsilon_\infty V+\frac{4\pi\hbar c\psi_1}{\tilde{g}^2}n\tilde{T}^2\frac{d}{d\tilde{T}}\ln\left[\widetilde{E^{3/2}_{\rm mp}}(\tilde{T})\right]+\frac{3}{2}nkT, \tag{91}$$

where $\beta=1/(kT)$. The first two terms on the right-hand side of this expression are the corrections occurring when the internal structure of the monopole is taken into account. Fig. 10 shows the dependence of $\tilde{T}^2\frac{\partial}{\partial\tilde{T}}\ln\left[\widetilde{E^{3/2}_{\rm mp}}\right]$ on the temperature $\tilde{T}$, and also the third term from the right-hand side of (91) for comparison.

6. ψ_1 AND $\Lambda_{\rm QCD}$

In our calculations, there is one undetermined quantity ψ_1 having the dimension of m^{-1}. One can assume that it might be relevant to some physical quantity known from QCD. Such a quantity is $\Lambda_{\rm QCD}=200$ MeV, which in units of m^{-1} is $\Lambda_{\rm QCD}\approx 10^{15}\ \mathrm{m}^{-1}$. This enables us to assume that

$$4\pi\hbar c\psi_1\approx\Lambda_{\rm QCD}. \tag{92}$$

In this case, the quantum corrections $(\mu^2)^{ab\mu\nu}$ and M^2 in Eqs. (31) and (32) can be written as

$$\left(\mu^2\right)^{ab\mu\nu}=\left(\frac{\Lambda_{\rm QCD}}{\hbar c}\right)^2\left(\tilde{\mu}^2\right)^{ab\mu\nu},\ \ M=\frac{\Lambda_{\rm QCD}}{g\hbar c}\tilde{M},$$

where $\tilde{\mu}$ and $\tilde{M}$ are dimensionless. The relation (92) also enables us to assert that $\Lambda_{\rm QCD}$ describes the dispersion $\left\langle\hat{A}^{m\mu}(y)\hat{A}^{n\nu}(x)\right\rangle$ of quantum fluctuations of the coset fields $\hat{A}^{m\mu}$.

Then, using $\Lambda_{\rm QCD}$, the thermodynamic formulas (86), (90), and (91) can be rewritten in the following manner:

$$F_{\rm gas}(T)=\epsilon_\infty V-n\frac{\Lambda_{\rm QCD}}{\tilde{g}^2}\ln\left[\overline{E^{3/2}_{\rm mp}}(T)\right]-nkT\ln\frac{V}{V_0}-\frac{3}{2}nkT\ln\frac{T}{T_0}, \tag{93}$$

$$U_{\rm gas}(T)=\epsilon_\infty V+\frac{\Lambda_{\rm QCD}}{\tilde{g}^2}n\tilde{T}^2\frac{\partial}{\partial\tilde{T}}\ln\left[\widetilde{E^{3/2}_{\rm mp}}(\tilde{T})\right]+\frac{3}{2}nkT, \tag{94}$$

$$\overline{E^{3/2}_{\rm mp}}\Big(T\Big)=\left(\frac{\Lambda_{\rm QCD}}{\tilde{g}^2}\right)^{3/2}\widetilde{E^{3/2}_{\rm mp}}(\tilde{T}). \tag{95}$$

7. CONCLUSION

In this paper, we investigate quantization of physical systems in which no small parameter exists that could be used to apply the standard perturbative methods. In particular, we propose a definition of nonperturbative quantum states in terms of Green functions by using the nonperturbative quantization procedure proposed and developed originally by Heisenberg. We first examine the operator field equations and mention that, due to their mathematical complexity, it is necessary to consider the equivalent system of differential equations that determine the infinite set of Green functions. *Since the general solution of such a system leads to the entire set of Green functions which must contain all quantum information about the system, we conclude that we can identify nonperturbative states with Green functions*. We also analyze the case of gauge fields.

The corresponding infinite set of equations for Green functions cannot be solved in general. However, we use earlier results according to which the infinite set of equations can be reduced to a set of two equations whose solutions describe flux tubes corresponding to realistic physical configurations of gauge fields. As a particular example, we study the thermodynamics of a quantum monopole and of a gas of noninteracting quantum monopoles embedded in the condensate.

Working in the framework of the two-equation approximation method for nonperturbative quantization à la Heisenberg, we have found the energy spectrum of one quantum monopole, using which the partition function and the corresponding thermodynamic quantities have been computed. Then we have calculated the partition function and thermodynamic quantities for a system of noninteracting quantum monopoles embedded in the condensate. It was shown that all thermodynamic quantities have quantum corrections related to the internal structure of the monopole. Physically this means that in the presence of fluctuations the energy received from a thermostat changes not only the kinetic energy of the monopoles but their internal energy as well. This is responsible for changes in the pressure, internal energy, etc. of a gas of monopoles.

It is important to note that the calculation of statistical and thermodynamic quantities for nonlinear theories quantized by using the nonperturbative methods à la Heisenberg leads to finite results, just as it is in quantum electrodynamics. A fundamental difference is that in the case considered in the present paper the calculations are carried out for nonperturbatively quantized fields which are not associated with particles (quanta) but are more like a turbulent fluid where at each point there are fluctuating velocities, pressures, etc.

In summary, we have

- formulated the notion of a nonperturbative quantum state;
- approximately defined nonperturbative quantum states for a quantum monopole and for a flux tube determined by the 2-point Green functions $G^{mn\mu\nu}$, $G^{ab\mu\nu}$ and the 4-point Green function $G^{mnpq}{}_{\mu\nu\rho\sigma}(x,y,z,u)$;
- numerically calculated the energy spectra for a single quantum monopole and for a flux tube;
- numerically calculated the partition function and thermodynamic quantities for a quantum monopole and for a dilute gas of noninteracting quantum monopoles.

The physical interpretation of the obtained results is that, using nonperturbative quantization, we have shown that nonperturbative vacuum in pure gluodynamics (without quarks) consists of a quantum condensate of the coset fields filled with quantum monopoles. This result confirms to the hypothesis proposed earlier [5–7] and confirmed by lattice calculations that the QCD vacuum is filled with monopoles [8].

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