ISSN 0202-2893, Gravitation and Cosmology, 2018, Vol. 24, No. 3, pp. 267–273. © Pleiades Publishing, Ltd., 2018.

# Topologically Nontrivial Solution in Einstein-Dirac Gravity on the Hopf Bundle

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Received August 23, 2017

**Abstract**—A topologically nontrivial solution in Einstein-Dirac gravity with a cosmological constant is obtained. The space-time has the Hopf bundle as a spatial section. It is shown that the Hopf invariant is related to the spinor current density. Two Dirac spinors are used for obtaining a diagonal energy-momentum tensor. Solutions to the nongravitating Dirac equation in background Lorentzian space-time with the Hopf bundle as a spatial section are also obtained. Nongravitating solutions of the Dirac equation are characterized by two quantum half-integer numbers $m, n$.

**DOI:** 10.1134/S0202289318030052

## 1. INTRODUCTION

In general relativity, there are a lot of solutions with gravitating fundamental fields: scalar, electromagnetic, and non-Abelian ones. However, very little is known about solutions in Einstein-Dirac gravity. For example, we do not know any asymptotically flat solution with a gravitating spinor field. Perhaps the problem here is related to the fact that the spinor field has a spin. As a result, the energy-momentum tensor for a spinor field has nondiagonal components. and not only $T_{t\varphi}$, as it happens for the Kerr metric.

Gravitating spinors with nonlinear self-interactions are well investigated in cosmology [1–5]. These papers study the role of a spinor field in considering the evolution of the anisotropic Universe described by Bianchi type VI, $\mathrm{VI}_0$, V, III, I, or isotropic Friedmann-Robertson-Walker (FRW) models. In [6, 7] models of the Universe with tachyonic and fermionic fields interacting through a Yukawa-type potential are investigated. In [8], a class of exact cosmological solutions with a neutral scalar field and a Majorana fermion field is found. A Dirac spinor in $D = 3$ dimensions coupled to topologically massive gravity is investigated in [9]. In [10], a mechanism is analyzed where quantum oscillations of the Dirac wave functions can prevent the formation of the big bang or big crunch singularity.

A simpler problem is that of seeking solutions in the background of a curved space-time. In the textbook [11], Dirac's equation in the Kerr background geometry is considered. In [12], topologically trivial solutions to the Dirac equation are obtained on a 3D sphere $S^3$.

Here we will consider two topologically nontrivial Dirac spinors coupled to Einstein gravity with the cosmological constant. The 3D section of the space-time metric is a Hopf bundle. The topological nontriviality means that the current of a spinor field is connected with the Hopf invariant. We consider two Dirac spinors. The energy-momentum tensor for each of them has nondiagonal components, related to the fact that the spinor has a spin. With our choice of the spinors, the total energy-momentum tensor has only diagonal components.

We will also investigate the nongravitating Dirac equation. Some special solutions will be found for the case where a Dirac spinor can be decoupled to Weyl spinors (this is, the case where the mass of a spinor is zero, $m = 0$). A numerical solution for a nonzero mass $m \neq 0$ will also be presented.

## 2. EINSTEIN-DIRAC GRAVITY

The Einstein-Dirac equations for two gravitating spinor fields $\psi_{1,2}$ are

$$R_{ab} - \frac{1}{2}\eta_{ab}R - \eta_{ab}\Lambda = \varkappa T_{ab}, \qquad (1)$$

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$$\left(i\gamma^\mu D_\mu - \tilde{\mu}_{1,2}\right)\psi_{1,2} = 0. \tag{2}$$

Here Latin indices $a, b = 0, 1, 2, 3$ are tetrad ones, Greek letters $\mu = 0, 1, 2, 3$ are space-time indices; $D_\mu = \partial_\mu + \frac{1}{4}\omega_{ab\mu}\gamma^a\gamma^b$ is the spinor covariant derivative; $\omega_{ab\mu}$ is the spin connection; the energy-momentum tensors for two spinor fields $\psi_{1,2}$ are

$$T_{\mu\nu} = T_{1;\mu\nu} + T_{2;\mu\nu}, \tag{3}$$

$$T_{1,2;\mu\nu} = \frac{i}{4}(\bar{\psi}_{1,2}\gamma_\mu D_\nu\psi_{1,2} - D_\nu\bar{\psi}_{1,2}\gamma_\mu\psi_{1,2} + \bar{\psi}_{1,2}\gamma_\nu D_\mu\psi_{1,2} - D_\mu\bar{\psi}_{1,2}\gamma_\nu\psi_{1,2}). \tag{4}$$

The spin connection $\omega_{ab\mu}$, the Ricci coefficients $\Delta_{\alpha\beta\gamma}$, and the anholonomy coefficients $\Sigma^a{}_{\mu\nu}$ are defined as (here we follow the textbook [13])

$$\omega_{ab\mu} = -e_a{}^\alpha e_b{}^\beta \Delta_{\alpha\beta\mu}, \tag{5}$$

$$\Delta_{\alpha\beta\gamma} = e_{a\alpha}\Sigma^a{}_{\beta\gamma} - e_{a\beta}\Sigma^a{}_{\alpha\gamma} - e_{a\gamma}\Sigma^a{}_{\alpha\beta}, \tag{6}$$

$$\Sigma^a{}_{\mu\nu} = \frac{1}{2}\left(\partial_\nu e^a{}_\mu - \partial_\mu e^a{}_\nu\right). \tag{7}$$

The Dirac matrices $\gamma^a$ in flat space are

$$\gamma^a = \left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -\sigma^j\\ \sigma^j & 0\end{pmatrix}\right\}, \quad j = 1, 2, 3, \tag{8}$$

where $\sigma^j$ are the Pauli matrices

$$\sigma^j = \left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\}. \tag{9}$$

## 3. THE SOLUTION

We seek a solution to the Einstein-Dirac equations (1) and (2) in the following form:

$$ds^2 = dt^2 - \frac{r^2}{4}\left[\left(d\chi^2 - \cos\theta d\varphi\right)^2 + d\theta^2 + \sin^2\theta d\varphi^2\right], \tag{10}$$

$$\psi_1 = e^{i\tilde{\Omega}_1 t}e^{in_1\chi}e^{im_1\varphi}\begin{pmatrix}\Theta_1(\theta)\\ \Theta_2(\theta)\\ \Theta_3(\theta)\\ \Theta_4(\theta)\end{pmatrix}, \tag{11}$$

$$\psi_2 = e^{i\tilde{\Omega}_2 t}e^{in_2\chi}e^{im_2\varphi}\begin{pmatrix}\Sigma_1(\theta)\\ \Sigma_2(\theta)\\ \Sigma_3(\theta)\\ \Sigma_4(\theta)\end{pmatrix}, \tag{12}$$

where the spatial metric

$$dl^2 = \frac{r^2}{4}\Big[\left(d\chi^2 - \cos\theta d\varphi\right)^2 + d\theta^2 + \sin^2\theta d\varphi^2\Big] \tag{13}$$

is the metric on the Hopf bundle, and $r$ is the radius of 3D sphere. The Hopf bundle can be presented as a $S^3$ sphere with topological mapping $S^3 \to S^2$, where the fibre $S^1$ is spanned by the coordinate $\psi$, and the metric on the base of the bundle $S^2$ is $dl^2 = (r/2)^2(d\theta^2 + \sin^2\theta d\varphi^2)$.

To calculate all these quantities, we must define a tetrad $e^a{}_\mu$ for the metric (10):

$$e^a{}_\mu = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{r}{2} & 0 & -\frac{r}{2}\cos\theta\\ 0 & 0 & \frac{r}{2} & 0\\ 0 & 0 & 0 & \frac{r}{2}\sin\theta\end{pmatrix}. \tag{14}$$

### *3.1. Dirac equations on the Hopf bundle*

The spinors (12) may transform under rotation at an angle $2\pi$ as follows:

$$\psi_{1,2}(\chi + 2\pi) = \pm\psi_{1,2}(\chi), \tag{15}$$

$$\psi_{1,2}(\varphi + 2\pi) = \pm\psi_{1,2}(\varphi). \tag{16}$$

Taking into account the exponentials $e^{in_{1,2}\chi}, e^{im_{1,2}\varphi}$ from (12), we see that the numbers $m_{1,2}, n_{1,2}$ should satisfy the condition

$$m_{1,2} + n_{1,2} = \pm\frac{N}{2}, \tag{17}$$

where $N$ is an integer.

After substituting the *Ansatz* (12) into the Dirac equation (2), we have

$$\Theta_1' + \Theta_1\left(\frac{\cot\theta}{2} + n\right) + \Theta_2\left(\frac{1}{4} - \frac{\Omega}{2} - n\cot\theta - \frac{m}{\sin\theta}\right) - \frac{\mu}{2}\Theta_4 = 0, \tag{18}$$

$$\Theta_2' + \Theta_2\left(\frac{\cot\theta}{2} - n\right) + \Theta_1\left(-\frac{1}{4} + \frac{\Omega}{2} - n\cot\theta - \frac{m}{\sin\theta}\right) + \frac{\mu}{2}\Theta_3 = 0, \tag{19}$$

$$\Theta_3' + \Theta_3\left(\frac{\cot\theta}{2} + n\right) + \Theta_4\left(\frac{1}{4} + \frac{\Omega}{2} - n\cot\theta - \frac{m}{\sin\theta}\right) + \frac{\mu}{2}\Theta_2 = 0, \tag{20}$$

$$\Theta_4' + \Theta_4\left(\frac{\cot\theta}{2} - n\right) + \Theta_3\left(-\frac{1}{4} - \frac{\Omega}{2} - n\cot\theta - \frac{m}{\sin\theta}\right) - \frac{\mu}{2}\Theta_1 = 0, \tag{21}$$

where, for brevity, we have omitted the index: $\Omega = \Omega_1$, $m, n = m_1, n_1$; $\Omega = r\tilde{\Omega}$ and $\mu = r\tilde{\mu}$. For $\Sigma_{1,2,3,4}$ we also have the same equations. Some special solutions to this set of equations are presented in Appendices A and B.

### 3.2. Energy-Momentum Tensor of Massless Spinor Fields, $m = 0$

To solve the Einstein-Dirac equations (1) and (2), we will use the following *Ansätze* for the spinors $\psi_{1,2}$:

$$\psi_{1,2} = e^{i\tilde{\Omega}_{1,2}t}e^{i\chi/2}\tilde{\Theta}_{1,2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \quad \tilde{\Omega}_{1,2} = \frac{3}{2r},$$

$$T_{1,2;ab} = \frac{\tilde{\Theta}^2_{1,2}}{r}\begin{pmatrix}3 & -2 & 0 & 0\\ -2 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}; \tag{22}$$

$$\psi_{1,2} = e^{i\tilde{\Omega}_{1,2}t}e^{-i\chi/2}\tilde{\Theta}_{1,2}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \quad \tilde{\Omega}_{1,2} = \frac{3}{2r},$$

$$T_{1,2;ab} = \frac{\tilde{\Theta}^2_{1,2}}{r}\begin{pmatrix}3 & 2 & 0 & 0\\ 2 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}; \tag{23}$$

$$\psi_{1,2} = e^{i\tilde{\Omega}_{1,2}t}e^{i\chi/2}\tilde{\Theta}_{1,2}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \quad \tilde{\Omega}_{1,2} = -\frac{3}{2r},$$

$$T_{1,2;ab} = \frac{\tilde{\Theta}^2_{1,2}}{r}\begin{pmatrix}-3 & -2 & 0 & 0\\ -2 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}; \tag{24}$$

$$\psi_{1,2} = e^{i\tilde{\Omega}_{1,2}t}e^{-i\chi/2}\tilde{\Theta}_{1,2}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}, \quad \tilde{\Omega}_{1,2} = -\frac{3}{2r},$$

$$T_{1,2;ab} = \frac{\tilde{\Theta}^2_{1,2}}{r}\begin{pmatrix}-3 & 2 & 0 & 0\\ 2 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}. \tag{25}$$

Here $\tilde{\Theta}_{1,2}$ are constants.

### 3.3. Solution of the Einstein–Dirac Equations

The Einstein tensor $G_{ab} = R_{ab} - (1/2)\eta_{ab}R$ for the tetrad (14) is

$$G_{ab} = \frac{1}{r^2}\mathrm{diag}(3, -1, -1, -1). \tag{26}$$

To have the same structure on the right-hand side of the Einstein equations [as in (26)], we have to use a sum of the energy-momentum tensors from (22) and (23). In this case the Einstein equations are

$$\frac{3}{r^2} - \Lambda = 48\pi l_{\mathrm{Pl}}^2\frac{\tilde{\Theta}^2}{r}, \tag{27}$$

$$-\frac{1}{r^2} + \Lambda = 16\pi l_{\mathrm{Pl}}^2\frac{\tilde{\Theta}^2}{r}; \tag{28}$$

here $\tilde{\Theta} = \tilde{\Theta}_{1,2}$, and $l_{\mathrm{Pl}}$ is the Planck length. The solution of these equations in a dimensionless form is

$$\tilde{\Theta}l_{\mathrm{Pl}}^{3/2} = \left(\frac{1}{32\pi}\frac{l_{\mathrm{Pl}}}{r}\right)^{1/2}, \tag{29}$$

$$\Lambda = 3/(2r^2). \tag{30}$$

It is useful to give some numerical estimate for this solution. For example, if we take $r \approx \Lambda_{\mathrm{obs}}^{-1/2} \approx 10^{26}$ m, then

$$\tilde{\Theta}l_{\mathrm{Pl}}^{3/2} \approx \left(\frac{1}{32\pi}l_{\mathrm{Pl}}\Lambda_{obs}^{1/2}\right) \approx 10^{-31}, \tag{31}$$

where $\Lambda_{\mathrm{obs}}$ is the observed value of the cosmological constant.

Finally, the solution of the Einstein-Dirac equations (1) and (2) is

$$ds^2 = dt^2 - \frac{r^2}{4}\left[(d\chi^2 - \cos\theta d\varphi)^2 + d\theta^2 + \sin^2\theta d\varphi^2\right], \tag{32}$$

$$\psi_1 = e^{i\tilde{\Omega}t}e^{i\chi/2}\tilde{\Theta}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \tag{33}$$

$$\psi_2 = e^{i\tilde{\Omega}t}e^{-i\chi/2}\tilde{\Theta}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \quad \tilde{\Omega} = \frac{3}{2r}, \tag{34}$$

with $\tilde{\Theta}$, $\epsilon$ from (29) and (30), and $r$ is the radius of a 3D sphere that is the total space of the Hopf bundle.

### 3.4. The Current and the Hopf Invariant

The covariant current for the spinor $\psi_1$ is

$$j_\mu = \bar{\psi}_1\gamma_\mu\psi_1. \tag{35}$$

Calculations with the spinor (12) give us

$$j_\mu = \Big\{\Theta_1^2 + \Theta_2^2 + \Theta_3^2 + \Theta_4^2, r(\Theta_1\Theta_2 - \Theta_3\Theta_4), 0, \frac{r}{2}[\sin\theta(\Theta_1^2 - \Theta_2^2 - \Theta_3^2 + \Theta_4^2) - 2\cos\theta(\Theta_1\Theta_2 - \Theta_3\Theta_4)]\Big\}. \tag{36}$$

A similar result can be obtained for the spinor $\psi_2$ by changing $\Theta \to \Sigma$. For the solution (33), we have

$$j_\mu = 2\tilde{\Theta}^2\Big\{1, \frac{r}{2}, 0, -\frac{r}{2}\cos\theta\Big\}, \tag{37}$$

The Hopf bundle with the metric (13) has the Hopf invariant defined as

$$H = \frac{1}{V}\int \Upsilon \wedge d\Upsilon\ , \tag{38}$$

where $\Upsilon$ is some 1-form which can be related to the spatial part of the covariant current (37), and $V = \int \sin\theta d\chi d\theta d\varphi$ is the volume of a $S^3$ sphere of unit radius. We construct the 1-form $\Upsilon$ from the spatial part of the current (37):

$$\Upsilon = \frac{1}{r\tilde{\Theta}^2}j_i dx^i = d\chi - \cos\theta d\varphi, \tag{39}$$

where $i = \chi, \theta, \varphi$. Substitution of $\Upsilon$ into (38) gives

$$H = 1. \tag{40}$$

We see that the topological nontriviality of the solution (33) is connected with the Hopf invariant (40).

### 3.5. Comparison with the Taub-NUT and Friedmann Solutions

Let us compare the solution (32)–(34) with the Taub-NUT solution

$$ds^2 = \frac{dt^2}{U(t)} - 4l^2U(t)(d\chi + \cos\theta d\phi^2) - (t^2 + l^2)(d\theta^2 + \sin^2\theta d\phi^2), \tag{41}$$

where $m$ and $l$ are constants and

$$U(t) = \frac{-t^2 + 2mt + l^2}{t^2 + l^2}. \tag{42}$$

The Taub-NUT metric describes empty space-time with a nontrivial topology: the spatial section is a 3D sphere $S^3$, that is, the total space of the Hopf bundle. 2D space with the metric $d\theta^2 + \sin^2\theta d\phi^2$ is the metric on the base of the bundle, and the coordinate $\chi$ spans the fiber $S^1$.

We can interpret the solution (29), (30), (32)–(34) as Taub-NUT space-time filled with the spinor fields $\psi_{1,2}$, satisfying the Dirac equations (2). Physically, this means that empty nontrivial Taub-NUT space-time with the Hopf bundle as the spatial section has a nontrivial evolution in time. But if we fill space-time with two spinors $\psi_{1,2}$ and a cosmological constant, then space-time becomes static.

The FRW metric is

$$ds^2 = dt^2 - a^2(t)dS_3^2, \tag{43}$$

where $dS_3^2$ is the metric on a 3D sphere $S_3$. $dS_3^2$ can be written in the standard way $dS_3^2 = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\varphi^2)$ or as the metric (13) on the Hopf bundle.

The metric (43) describes a Friedmann universe filled with matter. The Universe is not static and evolves from an initial singularity to a maximum size and then to the final singularity. As we see from the solution (29), (30), (32)–(34), if the Friedmann universe is filled with two spinor fields $\psi_{1,2}$ instead of matter plus a cosmological constant, then the universe becomes static without any time evolution.

## 4. DISCUSSION AND CONCLUSIONS

We have considered Einstein-Dirac gravity with a cosmological constant and obtained a topologically nontrivial solution for two gravitating Dirac spinors. We have used two spinors and some special choice of quantum numbers $m$, $n$ in order to obtain a diagonal energy-momentum tensor. The latter allows us to derive the required solution, which:

- is not asymptotically flat;
- has no event horizon;

- is topologically nontrivial since it is defined on the Hopf bundle, and the Hopf invariant is related to the current of the spinor field;
- cannot describe any quantum particle because of nonlinearity of the Einstein-Dirac equations: the spinor cannot be normalized to unity;
- can be regarded as Taub-NUT space-time filled with a spinor field + $\Lambda$;
- can be regarded as a Friedmann universe filled with two spinor fields (instead of matter) without time evolution.

*Appendix*

## A: SOLUTIONS OF THE NONGRAVITATING DIRAC EQUATION

*A1:* $\Theta_4 = \Theta_3 = 0$, $\mu = 0$

In this case we have the Dirac equations (18) and (19) in the form

$$\Theta_1' + \Theta_1\left(\frac{\cot\theta}{2} + n\right) + \Theta_2\left(\frac{1}{4} - \frac{\Omega}{2} - n\cot\theta - \frac{m}{\sin\theta}\right) = 0, \quad \text{(A.1)}$$

$$\Theta_2' + \Theta_2\left(\frac{\cot\theta}{2} - n\right) + \Theta_1\left(-\frac{1}{4} + \frac{\Omega}{2} - n\cot\theta - \frac{m}{\sin\theta}\right) = 0. \quad \text{(A.2)}$$

The solution is sought for in the form

$$\Theta_2 = \pm\Theta_1 = C\sin^\alpha\left(\frac{\theta}{2}\right)\cos^\beta\left(\frac{\theta}{2}\right), \quad \text{(A.3)}$$

where $C$ is an arbitrary constant. Substituting this into Eqs. (A.1) and (A.2), we have the following solutions:

$$\Omega/2 - 1/4 = \pm n, \quad \text{(A.4)}$$

$$\alpha = \pm(n+m) - 1/2, \quad \text{(A.5)}$$

$$\beta = \pm(n-m) - 1/2. \quad \text{(A.6)}$$

Regular solutions for $\Theta_{1,2}$ and for $\theta \in (0,\pi)$ do exist for $\alpha \geq 0, \beta \geq 0$. This means that we have the following restrictions on the quantum numbers $m, n$:

$$\pm(n+m) \geq 1/2, \quad \text{(A.7)}$$

$$\pm(n-m) \geq 1/2. \quad \text{(A.8)}$$

*A2:* $\Theta_1 = \Theta_2 = 0$, $\mu = 0$

In this case we have the Dirac equations (20) and (21) in the form

$$\Theta_3' + \Theta_3\left(\frac{\cot\theta}{2} + n\right) + \Theta_4\left(\frac{1}{4} + \frac{\Omega}{2} - n\cot\theta - \frac{m}{\sin\theta}\right) = 0, \quad \text{(A.9)}$$

$$\Theta_4' + \Theta_4\left(\frac{\cot\theta}{2} - n\right) + \Theta_3\left(-\frac{1}{4} - \frac{\Omega}{2} - n\cot\theta - \frac{m}{\sin\theta}\right) = 0. \quad \text{(A.10)}$$

The solution is sought for in the form

$$\Theta_4 = \pm\Theta_3 = C\sin^\alpha\left(\frac{\theta}{2}\right)\cos^\beta\left(\frac{\theta}{2}\right), \quad \text{(A.11)}$$

where $C$ is an arbitrary constant. Substituting this into Eqs. (18)–(21), we have the following solutions:

$$\Omega/2 + 1/4 = \mp n, \quad \text{(A.12)}$$

$$\alpha = \pm(n+m) - 1/2, \quad \text{(A.13)}$$

$$\beta = \pm(n-m) - 1/2. \quad \text{(A.14)}$$

Once again, for regularity of the functions $\Theta_{3,4}$ it is necessary to have the following restrictions on $m, n$:

$$\pm(n+m) \geq 1/2, \quad \text{(A.15)}$$

$$\pm(n-m) \geq 1/2. \quad \text{(A.16)}$$

## B: SOLUTIONS OF THE NONGRAVITATING DIRAC EQUATION WITH $\mu \neq 0$

Substituting $\Theta_{3,4}$ from (18) and (19) into (20)-(21), we have the following set of equations:

$$\Theta_1'' + \cot\theta\Theta_1' + \frac{\Theta_2'}{2} + \Theta_1\left[\frac{\cot^2\theta}{4} - \frac{1}{2\sin^2\theta} - \left(n\cot\theta + \frac{m}{\sin\theta}\right)^2 - \left(\frac{n}{2}\cot\theta + \frac{1}{2}\frac{m}{\sin\theta}\right)\right] + \Theta_2\left(-\frac{n}{2} + m\frac{\cot\theta}{\sin\theta} + \frac{n}{\sin^2\theta} + \frac{\cot\theta}{4}\right) = \alpha\Theta_1, \quad \text{(B.1)}$$

$$\Theta_2'' + \cot\theta\Theta_2' - \frac{\Theta_1'}{2} + \Theta_2\left[\frac{\cot^2\theta}{4} - \frac{1}{2\sin^2\theta} - \left(n\cot\theta + \frac{m}{\sin\theta}\right)^2 + \left(\frac{n}{2}\cot\theta + \frac{1}{2}\frac{m}{\sin\theta}\right)\right] + \Theta_1\left(-\frac{n}{2} + m\frac{\cot\theta}{\sin\theta} + \frac{n}{\sin^2\theta} - \frac{\cot\theta}{4}\right) = \alpha\Theta_2, \quad \text{(B.2)}$$

$$\alpha = \frac{1}{16} + \mu^2 - \frac{\Omega^2}{4} + n^2. \qquad \text{(B.3)}$$

If we consider Eq (B.1) for $\theta < 0$, then we see that there may exist solutions with $\Theta_1(-\theta) = \Theta_2(\theta)$, $\Theta_2'(-\theta) = -\Theta_1'(\theta)$, and $\Theta_2''(-\theta) = \Theta_1''$ may exist. Then we can write one nonlocal equation

$$\Theta_1''(\theta) + \cot\theta\,\Theta_1'(\theta) - \frac{\Theta_1'(-\theta)}{2} + \Theta_1(\theta)\left[\frac{\cot^2\theta}{4} - \frac{1}{2\sin^2\theta} - \left(n\cot\theta + \frac{m}{\sin\theta}\right)^2 - \left(\frac{n}{2}\cot\theta + \frac{1}{2}\frac{m}{\sin\theta}\right)\right] + \Theta_1(-\theta)\left(-\frac{n}{2} + m\frac{\cot\theta}{\sin\theta} + \frac{n}{\sin^2\theta} + \frac{\cot\theta}{4}\right) = \alpha\Theta_1(\theta), \qquad \text{(B.4)}$$

and the same for $\Theta_2$. Numerical investigations confirm this statement.

Another way to solve Eqs. (B.1) and (B.2) is a decomposition of $\Theta_{1,2}$ to odd and even functions:

$$Y_1 = \Theta_1 + \Theta_2 \ \text{ is an even function}, \qquad \text{(B.5)}$$

$$Y_2 = \Theta_1 - \Theta_2 \ \text{ is an odd function}. \qquad \text{(B.6)}$$

For the functions $Y_1, Y_2$ we have the following equations:

$$Y_1'' + \cot\theta\, Y_1' + \frac{Y_2'}{2} + Y_1\left[\frac{\cot^2\theta}{4} - \frac{1}{2\sin^2\theta} - \left(n\cot\theta + \frac{m}{\sin\theta}\right)^2 + \frac{n}{2} - m\frac{\cot\theta}{\sin\theta} - \frac{n}{\sin^2\theta}\right] = Y_2\left(\frac{n}{2}\cot\theta + \frac{1}{2}\frac{m}{\sin\theta} - \frac{\cot\theta}{4}\right) = \alpha Y_1, \qquad \text{(B.7)}$$

$$Y_2'' + \cot\theta\, Y_2' - \frac{Y_1'}{2} + Y_2\left[\frac{\cot^2\theta}{4} - \frac{1}{2\sin^2\theta} - \left(n\cot\theta + \frac{m}{\sin\theta}\right)^2 - \frac{n}{2} + m\frac{\cot\theta}{\sin\theta} + \frac{n}{\sin^2\theta}\right] - Y_1\left(\frac{n}{2}\cot\theta + \frac{1}{2}\frac{m}{\sin\theta} + \frac{\cot\theta}{4}\right) = \alpha Y_2. \qquad \text{(B.8)}$$

We will solve Eqs. (B.1) and (B.2) as an eigenvalue problem with the eigenvalue $\alpha$ and eigenfunctions $\Theta_{1,2}$.

## NUMERICAL SOLUTION OF (B.1) AND (B.2)

To avoid a singularity in Eqs. (B.1) and (B.2) at the angle $\theta = 0$, we have to use the following form of the functions $\Theta_{1,2}$:

$$\Theta_1 = \frac{\alpha_2}{2}\theta^2 + \frac{\alpha_3}{6}\theta^3 + \ldots, \qquad \text{(B.9)}$$

$$\Theta_2 = \frac{\beta_2}{2}\theta^2 + \frac{\beta_3}{6}\theta^3 + \ldots. \qquad \text{(B.10)}$$

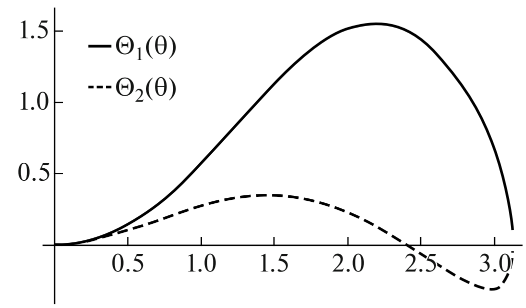


**Fig. 1.** The profiles of $\Theta_{1,2}(\theta)$ for equations (B.1) and (B.2).

Substituting (B.9) and (B.10) into (B.1) and (B.2), we have

$$\beta_2 = \pm\alpha_2, \qquad \text{(B.11)}$$

$$\beta_3 = \pm\alpha_3, \qquad \text{(B.12)}$$

$$m + n = \pm 3/2,\ \pm 5/2. \qquad \text{(B.13)}$$

Numerical solutions are presented in Fig. 1 for the following values of the parameters:

$$m + n = 5/2, \quad n = 1, \qquad \text{(B.14)}$$

$$\beta_2 = \alpha_2 = 1.0, \qquad \text{(B.15)}$$

$$\beta_3 = -\alpha_2 = -1.0. \qquad \text{(B.16)}$$

Numerical calculations give

$$\alpha = \frac{1}{16} + \mu^2 - \frac{\Omega^2}{4} + n^2 \approx -4.13208. \qquad \text{(B.17)}$$

## ACKNOWLEDGMENTS

This work was supported by Grant Φ.0755 in fundamental research in natural sciences by the MES of RK. I am very grateful to V. Folomeev for fruitful discussions and comments.

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