Wormhole solutions with a complex ghost scalar field and their instability

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We study compact configurations with a nontrivial wormholelike spacetime topology supported by a complex ghost scalar field with a quartic self-interaction. For this case, we obtain regular asymptotically flat equilibrium solutions possessing reflection symmetry. We then show their instability with respect to linear radial perturbations.

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I. INTRODUCTION

Despite the fact that at the moment there is only one experimentally observed elementary scalar particle—the Higgs boson—it is commonly believed that other types of fundamental scalar fields do also exist in nature. Such scalar fields are widely used on microscopic scales in constructing models of particle physics, they are often considered in modeling various types of compact astrophysical objects, and they are important ingredients in describing the cosmological evolution of the early and the present Universe. In particular, real and complex scalar fields are employed in constructing models of gravitating configurations—the so-called boson stars [1–4]. Depending on the masses and self-interactions of the scalar fields, the resulting boson stars may be microscopic; they may possess masses and sizes comparable to those of other compact objects, like neutron stars and black holes; or they may even be suitable to model the halos of galaxies.

When one considers renormalizable field theories of real scalar fields in Minkowski space, it is known that it is impossible to get static regular solutions, since in a flat four-dimensional spacetime Derrick's theorem applies [5,6]. Coupling the scalar fields to gravity does not change this basic finding. There are no localized static solutions for self-gravitating real scalar fields; i.e., there are no scalar *geons* [7]. (Note that we consider neither self-gravitating Skyrmions here, since the Skyrme model possesses a term quartic in the currents [8], nor static scalarons, which possess a not strictly positive scalar potential [9].)

However, the possibility of obtaining localized static nonsingular solutions arises when one involves the so-called ghost scalar fields, i.e., fields with the opposite sign in front of the kinetic term of the scalar field Lagrangian density. The possible existence of ghost scalar fields in nature is indirectly supported by the observed accelerated expansion of the present Universe (see, e.g., Refs. [10–12]). Namely, to explain the recent observational data [13], one should take the so-called exotic matter into consideration, the effective pressure of which is negative and the modulus of which is greater than its energy density.

One way to introduce such exotic matter is through the use of ghost scalar fields [14,15]. On relatively small scales comparable to the sizes of stars, such fields permit obtaining solutions of the Einstein-matter equations describing

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configurations with a nontrivial wormholelike topology. There are configurations supported by massless ghost scalar fields [16–18] and systems possessing a scalar field potential [19,20]. In addition, such fields allow one to get compact configurations with a trivial spacetime topology [21].

Here, we demonstrate the possibility of obtaining localized regular solutions with a nontrivial wormholelike topology supported by a complex ghost scalar field. To the best of our knowledge, previously, only real ghost scalar fields have been employed in modeling wormholelike systems. The use of the complex ghost scalar field allows one to introduce a harmonic time dependence analogously to the case of a complex canonical scalar field employed in the construction of boson stars. We also endow the complex ghost scalar field with a nontrivial quartic self-interaction potential.

The paper is organized as follows. In Sec. II A, the general set of equations is derived for the equilibrium configurations with a nontrivial spacetime topology supported by a complex ghost scalar field. We present the numerically obtained solutions for these configurations in Sec. II B. Subsequently, a linear stability analysis is performed for these solutions in Sec. III. Finally, in Sec. IV, we summarize the obtained results.

II. EQUILIBRIUM CONFIGURATIONS

A. General set of equations

We consider here a model of a gravitating complex ghost scalar field. We start from the action (hereafter, we work in units where $c = \hbar = 1$)

$$S=\int d^4x\sqrt{-g}\left[-\frac{1}{16\pi G}R+\frac{1}{2}[\varepsilon g^{\mu\nu}\partial_\mu\Phi^*\partial_\nu\Phi-V(|\Phi|^2)]\right]. \tag{1}$$

Here, Φ is a complex scalar field with the potential $V(|\Phi|^2)$, and $\varepsilon = +1$ or -1 corresponds to canonical or ghost fields, respectively. This action is invariant under a global phase transformation $\Phi \to e^{i\theta}\Phi$, which implies the conservation of its generator N corresponding to the total particle number.

By varying (1) with respect to the metric, one obtains the Einstein equations with the energy-momentum tensor

$$T^\mu_\nu = \frac{\varepsilon}{2}g^{\mu\sigma}(\partial_\sigma\Phi^*\partial_\nu\Phi+\partial_\sigma\Phi\partial_\nu\Phi^*) - \frac{1}{2}\delta^\mu_\nu(\varepsilon g^{\lambda\sigma}\partial_\lambda\Phi^*\partial_\sigma\Phi - V). \tag{2}$$

In turn, varying (1) with respect to the scalar field, one obtains the field equation for the scalar field Φ,

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left[\sqrt{-g}g^{\mu\nu}\frac{\partial\Phi}{\partial x^\nu}\right] = -\varepsilon\frac{dV}{d|\Phi|^2}\Phi. \tag{3}$$

Our aim here is to study equilibrium wormhole solutions and to consider their stability. For this purpose, we take the spherically symmetric metric in the general form

$$ds^2 = e^\nu(dx^0)^2 - e^\lambda dr^2 - e^\mu d\Omega^2, \tag{4}$$

where ν, λ, and μ are functions of the radial coordinate r and the time coordinate x^0 and $d\Omega^2$ is the metric on the unit two-sphere. When considering equilibrium wormholelike configurations, it is convenient to choose the metric in polar Gaussian coordinates

$$ds^2 = e^\nu(dx^0)^2 - dr^2 - R^2 d\Omega^2, \tag{5}$$

where now ν and $e^\mu = R^2$ are functions of r only.

In the case of canonical scalar fields (i.e., when $\varepsilon = +1$), one can obtain the well-known boson-star solutions which possess a trivial spacetime topology. (For a general overview on the subject of boson stars, see, e.g., the reviews [1–4].) Note that one can also obtain boson star–like objects with a nontrivial spacetime topology, when an additional real ghost scalar field is included [22,23].

Here, we study the case of localized solutions with a wormholelike topology, which is provided by the presence of a complex ghost scalar field, i.e., for $\varepsilon = -1$ in the action (1). Then, in order to have no time dependence in the Einstein equations, we employ for the complex ghost scalar field the harmonic ansatz

$$\Phi(x^0, r) = \phi(r)e^{-i\omega x^0}. \tag{6}$$

As in the case of boson stars, this ansatz ensures that the spacetime of the system under consideration remains static.

The above ansatz then leads to the following system of Einstein-scalar equations:

$$-\left[2\frac{R''}{R}+\left(\frac{R'}{R}\right)^2\right]+\frac{1}{R^2} = 8\pi G T^0_0 = 4\pi G[-(\phi'^2+\omega^2 e^{-\nu}\phi^2)+V], \tag{7}$$

$$-\frac{R'}{R}\left(\frac{R'}{R}+\nu'\right)+\frac{1}{R^2} = 8\pi G T^1_1 = 4\pi G(\phi'^2+\omega^2 e^{-\nu}\phi^2+V), \tag{8}$$

$$\frac{R''}{R}+\frac{1}{2}\frac{R'}{R}\nu'+\frac{1}{2}\nu''+\frac{1}{4}\nu'^2 = -8\pi G T^2_2 = -4\pi G(-\phi'^2+\omega^2 e^{-\nu}\phi^2+V), \tag{9}$$

$$\phi''+\left(\frac{1}{2}\nu'+2\frac{R'}{R}\right)\phi'+\left(\omega^2 e^{-\nu}+\frac{dV}{d|\Phi|^2}\right)\phi = 0. \tag{10}$$

Depending on the form of the potential V and the boundary conditions, one then obtains localized equilibrium solutions by solving this set of equations numerically.

Here, we would like to demonstrate that one can obtain wormholelike solutions supported by a complex ghost scalar field for a relatively simple potential containing the quartic term $|\Phi|^4$. Let us therefore recall that in the case of a real ghost scalar field Φ there are wormholelike regular solutions for a quartic potential of the type $V \sim -(1-\bar{\lambda}\Phi^2)^2$ [19,20]. In this case, the corresponding solutions for the scalar field start in one of the maxima of this potential ($\Phi_{\max} = -1/\sqrt{\bar{\lambda}}$) and end in the other maximum ($\Phi_{\max} = +1/\sqrt{\bar{\lambda}}$). However, when one generalizes this potential to a complex ghost scalar field, it is no longer possible to get regular solutions.

Hence, in order to obtain regular asymptotically flat solutions with a nontrivial topology, we choose another form of the quartic potential. Namely, we seek regular zero-node solutions for a scalar field the potential of which has the form

$$V = -m^2|\Phi|^2 + \frac{1}{2}\bar{\lambda}|\Phi|^4. \tag{11}$$

Here, m and $\bar{\lambda}$ are free parameters of the scalar field. Let us point out that this potential has also a reversed sign in front of the quadratic term as compared to a canonical mass term. When such a reversed sign appears for canonical scalar fields, this signals spontaneous symmetry breaking as in the case of the Mexican hat potential for the Higgs boson. Together with a canonical mass term, such a quartic self-interaction term has been used before when modelling boson stars [24].

For the time-dependent complex ghost scalar fields, the above potential (11) admits only topologically trivial solutions, which start and end at the same maximum, in contrast to the case studied in Refs. [19,20]. In the terminology of Lee and collaborators [2,6], such solutions are referred to as nontopological soliton solutions, since their existence is not based on a conserved topological current.

B. Numerical solutions

Let us now turn to the discussion of the numerical solutions. In order to solve the above set of equations numerically, it is convenient to introduce new dimensionless variables

$$x = mr, \qquad X = mR, \qquad \Omega = \omega/m,$$
$$\Lambda = \frac{\bar{\lambda}}{4\pi G m^2}, \qquad \varphi = \sqrt{4\pi G}\phi. \tag{12}$$

Then, using the potential (11), one can rewrite Eqs. (7)–(10) in the form

$$-\left[2\frac{X''}{X} + \left(\frac{X'}{X}\right)^2\right] + \frac{1}{X^2} = -\varphi'^2 - \Omega^2 e^{-\nu}\varphi^2 - \varphi^2 + \frac{\Lambda}{2}\varphi^4, \tag{13}$$

$$-\frac{X'}{X}\left(\frac{X'}{X} + \nu'\right) + \frac{1}{X^2} = \varphi'^2 + \Omega^2 e^{-\nu}\varphi^2 - \varphi^2 + \frac{\Lambda}{2}\varphi^4, \tag{14}$$

$$\frac{X''}{X} + \frac{1}{2}\frac{X'}{X}\nu' + \frac{1}{2}\nu'' + \frac{1}{4}\nu'^2 = \varphi'^2 - \Omega^2 e^{-\nu}\varphi^2 + \varphi^2 - \frac{\Lambda}{2}\varphi^4, \tag{15}$$

$$\varphi'' + \left(\frac{1}{2}\nu' + 2\frac{X'}{X}\right)\varphi' + (\Omega^2 e^{-\nu} - 1 + \Lambda\varphi^2)\varphi = 0. \tag{16}$$

Note that these equations are not all independent because of the Bianchi identities, so one can use any three of them in calculations. Here, we have solved the set of Eqs. (13), (15), and (16), treating the first-order equation (16) as a constraint equation to monitor the accuracy of the results, since it ought to be satisfied identically for any solution of the chosen set of equations.

When solving the above set of equations, we use the following symmetric boundary conditions given at the center $r = 0$,

$$X(0) = X_c, \qquad \nu(0) = \nu_c, \qquad \varphi(0) = \varphi_c, \tag{17}$$

with all first-order derivatives being equal to zero at the center. Then, the constraint equation (14) yields

$$X_c = \frac{1}{\varphi_c\sqrt{\Omega^2 e^{-\nu_c} - 1 + (\Lambda/2)\varphi_c^2}}. \tag{18}$$

Taking this expression into account and expanding the metric function X in the vicinity of the center as $X \approx X_c + 1/2 X_2 x^2$, one finds from Eq. (13) the following value for the second derivative of X at the center

$$X_2 = \Omega^2 e^{-\nu_c}\varphi_c^2 X_c,$$

which is always positive. This means that the quantity X_c corresponds to the radius of the wormhole throat (and not to the radius of an equator) residing at the center, since the wormhole throat is defined by $X_{\rm th} = \min\{X(x)\}$. Consequently, throughout the paper, we will deal only with configurations possessing a single wormhole throat.

So, for the systems under consideration, we have three parameters, φ_c, ν_c, and Ω, one of which can be chosen freely. For example, as in the case of boson stars [24], this free parameter can be chosen to be φ_c. Then, the other two parameters must be chosen in such a way as to provide asymptotic flatness of the spacetime, when φ, φ', and $\nu \to 0$ and $X \to x$. In this sense, we are dealing with an eigenvalue problem for the parameters ν_c and Ω. The corresponding asymptotic behavior of the solutions is as follows,

$$
\begin{aligned}
&e^{\nu} \to 1 - \frac{2C_2}{x}, \qquad X \to x, \qquad X' \to 1 - \frac{C_2}{x}, \\
&\varphi \to C_1 \exp(-\sqrt{1-\Omega^2}x)x^{\beta} \quad \text{for } 0 \le \Omega < 1, \\
&\varphi \to C_3 \frac{\exp(-\sqrt{8|C_2|x})}{x^{3/4}} \quad \text{for } \Omega = 1, \qquad (19)
\end{aligned}
$$

where C_1, C_2, and C_3 are integration constants and $\beta = -1 + C_2\Omega^2/\sqrt{1-\Omega^2}$. Note that at $\Omega \to 1$ the integration constant C_2, which corresponds to the Arnowitt–Deser–Misner (ADM) mass of the wormhole, is always negative for the configurations under consideration.

We have solved the set of Eqs. (13)–(16) numerically, together with the boundary conditions (17) and (18), varying the strength Λ of the self-interaction in the interval $1 \le \Lambda \le 20\,000$ and the value of the boson frequency Ω in the interval $0 \le \Omega \le 1$. We demonstrate a set of typical solutions for the scalar field function φ and the metric function ν in Fig. 1, where we display the solutions only in one asymptotically flat part of the spacetime, since the solutions are symmetric with respect to $x \to -x$. Here, four different values of Λ are chosen, $\Lambda = 1$, 3, 10, and 30.

The values of the boson frequency chosen in the plots, $\Omega = 0$ and $\Omega = 1$, represent the boundary values of the physically acceptable interval. Note that the limiting case $\Omega = 0$ corresponds to configurations supported by a real

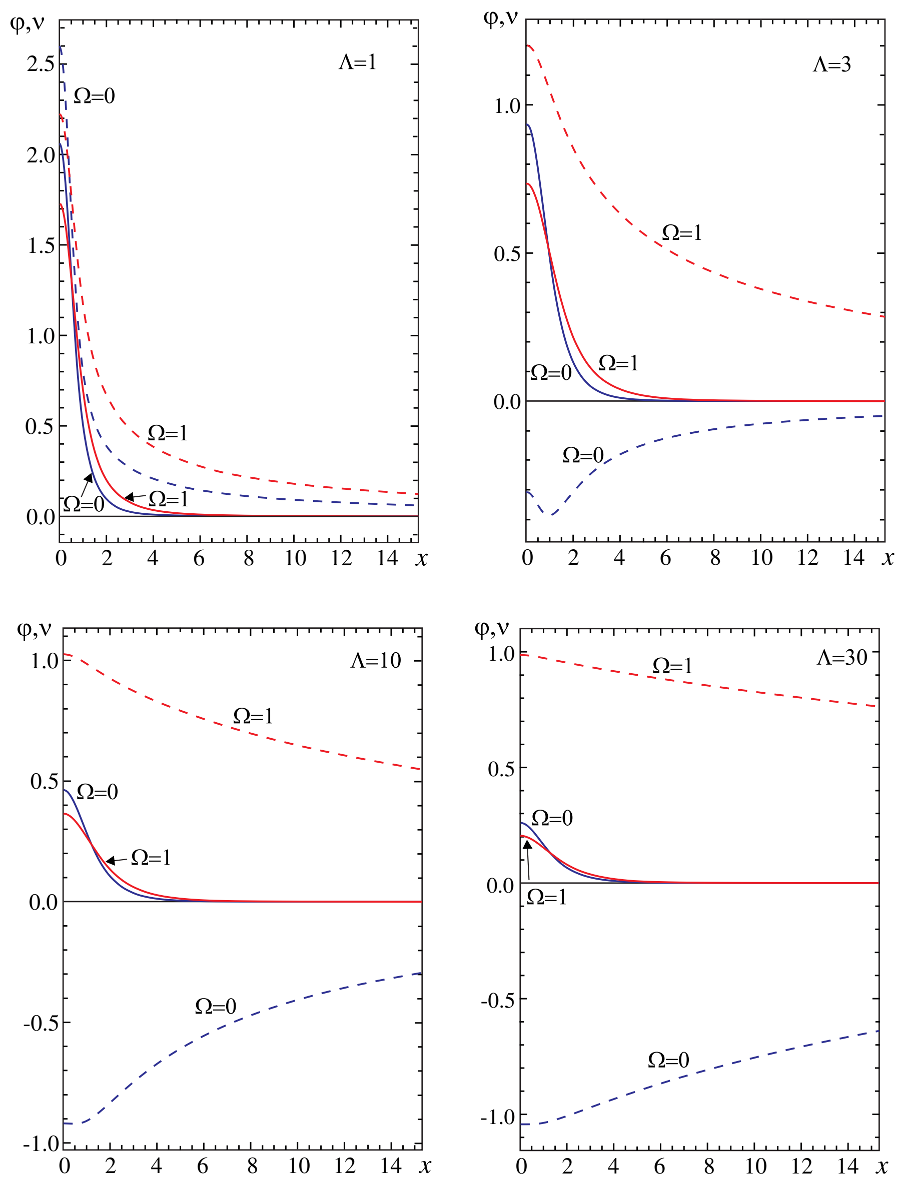

FIG. 1. Typical solutions for the scalar field function φ (solid lines) and the metric function ν (dashed lines) for different values of self-coupling constant Λ. (In view of the symmetry $x \to -x$, only the solutions for positive x are shown.) For the boson frequency Ω, the lower limit $\Omega = 0$ is chosen, while the upper limit is represented by the value $\Omega = 1$ in all graphs.

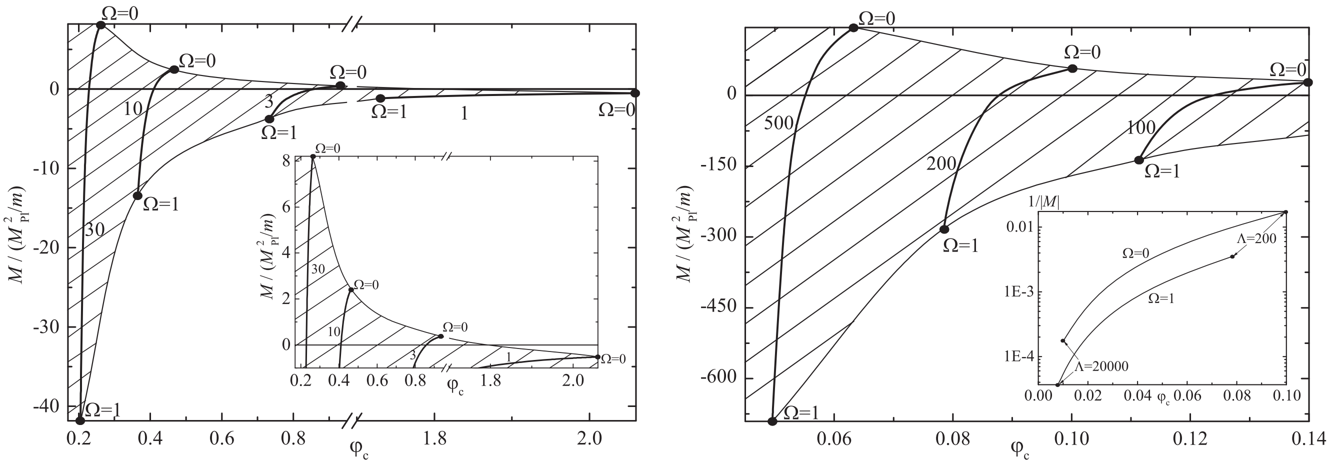

FIG. 2. The bold curves show the wormhole mass M as a function of the central value of the scalar field φ_c for different values of the self-coupling constant Λ (designated by the numbers near the curves). For other values of Λ, the masses lie within the shaded regions, enclosed by the limiting curves for the boson frequencies $\Omega = 0$ and $\Omega = 1$. In the left panel, the inset shows the region where the masses are close to zero. In the right panel, the inset demonstrates the growth of the modulus of the ADM mass as $\phi_c \to 0$, when $\Lambda \to \infty$.

ghost scalar field. For values of Ω in between these limiting values, the graphs of the above functions will lie between those shown in Fig. 1.

Let us now consider the ADM mass of the above systems. In the case of spherical symmetry, the Misner-Sharp [25] mass $M(r)$ associated with the volume enclosed by a sphere with circumferential radius R_c, corresponding to the center of the configuration, and another sphere with circumferential radius $R > R_c$ can be defined as follows:

$$M(r) = \frac{1}{2G} R_c + 4\pi \int_{R_c}^{r} T_0^0 R^2 dR. \qquad (20)$$

Taking the boundary to (spacelike) infinity, the Misner-Sharp mass leads to the ADM mass. In the dimensionless variables (12), this becomes

$$\mathcal{M}(x) \equiv \frac{M(x)}{M_{\mathrm{Pl}}^2/m} = \frac{1}{2}\left[X_c - \int_0^x \left(\varphi'^2 + \Omega^2 e^{-\nu}\varphi^2 + \varphi^2 - \frac{\Lambda}{2}\varphi^4\right) X^2 \frac{dX}{dx'} dx'\right]. \qquad (21)$$

The results of the calculations of the mass are shown in Fig. 2. It is interesting to compare these results with those obtained for systems with a trivial spacetime topology—boson stars (see, e.g., Refs. [1–4]). For the boson stars, solutions are sought for some range of central values of the scalar field φ_c, starting from zero, which leads to a set of configurations with different masses. The typical behavior of the dependence of the mass on the central value of the scalar field is as follows: As $\varphi_c \to 0$, the mass of the system goes to zero as well, and the boson frequency goes to its upper limit, $\Omega \to 1$. When φ_c increases, the mass at first also increases, reaching the maximum value at some critical φ_c^{cr}. Then, the mass decreases, reaches a local minimum, increases again, and exhibits a damped oscillation toward some limiting value at large φ_c, associated with a limiting boson frequency $\Omega \neq 0$.

In contrast to those systems with a trivial topology, the configurations with a nontrivial topology considered here possess the following distinctive features:

(1) When φ_c increases, there is a monotonic change of the mass (no extrema).

(2) Regular solutions exist for all physically acceptable values $0 \leq \Omega \leq 1$.

(3) As $\Lambda \to \infty$, the central value of the scalar field $\varphi_c \to 0$. In this case, both the size of the throat X_c and the ADM mass of the system increase without limit, as shown for the mass in the inset of the right panel of Fig. 2. (Note that the mass is positive for $\Omega \sim 0$ and large Λ and negative for $\Omega \to 1$.)

Figure 3 demonstrates how the dimensionless radius X_c of the wormhole throat, which also enters the equation for the mass (21), changes with the self-coupling constant Λ. Comparing Figs. 2 and 3, one notes that, despite the fact that with increasing Ω the radius of the throat increases, the ADM mass of the system decreases.

Let us finally calculate the total number of particles N forming the system. N can be obtained from the continuity equation $j^\mu_{;\mu} = 0$, where the four-current j^μ is given by

$$j^\mu = i g^{\mu\nu}(\partial_\nu \Phi \Phi^* - \partial_\nu \Phi^* \Phi).$$

This leads to the conserved charge

$$N = \int \sqrt{-g} j^0 d^3x = 8\pi\omega \int_0^r e^{-\nu/2} R^2 \phi^2 dr' = \frac{2\Omega}{(m/M_{\mathrm{Pl}})^2} \int_0^x e^{-\nu/2} X^2 \varphi^2 dx'. \qquad (22)$$

The results for the total particle number are shown in Fig. 4. It is seen that for a fixed finite value of the boson frequency

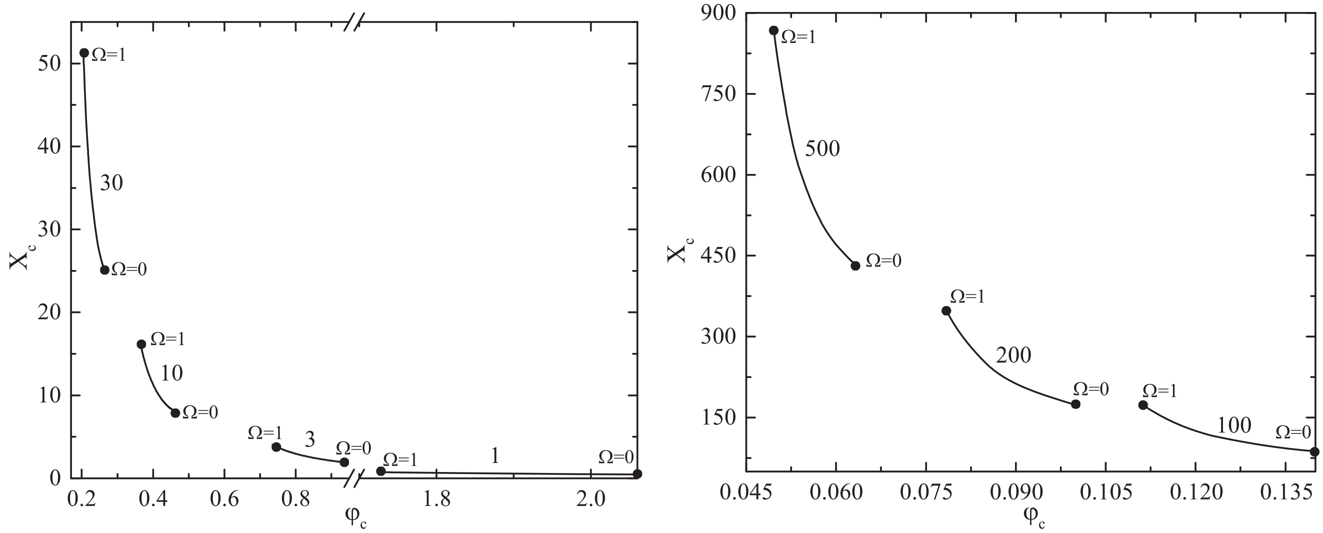

FIG. 3. The dimensionless radius X_c (18) of the wormhole throat is shown as a function of the central value of the scalar field φ_c for different values of the self-coupling constant Λ (designated by the numbers near the curves). The end points of the curves correspond to the boson frequencies $\Omega = 0$ and $\Omega = 1$.

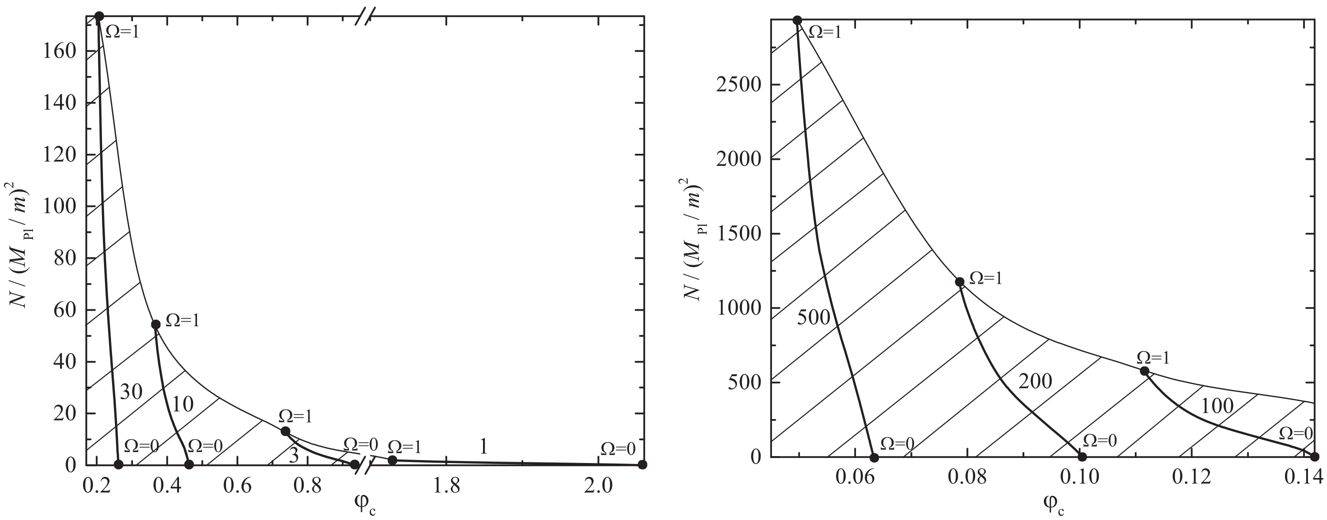

FIG. 4. The number of particles N as a function of the central value of the scalar field φ_c for different values of the self-coupling constant Λ (designated by the numbers near the curves). For other values of Λ, the number of particles lies within the shaded regions, enclosed by the limiting curves for the boson frequencies $\Omega = 0$ and $\Omega = 1$.

Ω the number of particles increases with increasing Λ. Note that the total particle number vanishes in the limit $\Omega = 0$. This corresponds to the fact that for a real scalar field there is no such conserved current.

III. LINEAR STABILITY ANALYSIS

In this section, we perform a linear stability analysis of the above equilibrium solutions. We derive the appropriate set of equations and then analyze them numerically. In doing so, we start from the stability analysis of the systems supported by a real scalar field; i.e., we first consider the case of $\Omega = 0$. Our aim here is to clarify how the generalization of these configurations to the case of a complex scalar field with $\Omega \neq 0$ influences their stability.

In performing the stability analysis, we start from the general metric (4). The perturbed solutions are then sought in the form

$$\nu = \nu_0(r) + \nu_p(r, x^0), \qquad \lambda = \lambda_0(r) + \lambda_p(r, x^0),$$
$$\mu = \mu_0(r) + \mu_p(r, x^0).$$

The subscript 0 denotes the static background solutions from the previous section, and the subscript p denotes the perturbations. Also, we represent the complex scalar field in terms of two distinct real functions, Φ_1 and Φ_2, retaining the harmonic time dependence of the unperturbed solutions as a factor

$$\Phi(r,x^0) = [\Phi_1(r,x^0) + i\Phi_2(r,x^0)]e^{-i\omega x^0}.$$

For the unperturbed system, $\Phi_1(r,x^0) \to \phi_0(r)$, and $\Phi_2 = 0$. The perturbative expressions for these scalar functions are then taken in the form

$$\Phi_1 = \phi_0(r) + \Phi_{1p}(r,x^0), \qquad \Phi_2 = \Phi_{2p}(r,x^0).$$

Using the above expressions, one can get the following perturbed components of the Einstein equations E_0^0, E_1^1, and E_2^2 (here and below, we take $\lambda_0 = 0$, since in obtaining the equilibrium solutions in Sec. II we have employed this gauge),

$$\mu_p'' + \frac{3}{2}\mu_0'\mu_p' - \frac{1}{2}\mu_0'\lambda_p' - \lambda_p\left(\mu_0'' + \frac{3}{4}\mu_0'^2\right) + e^{-\mu_0}\mu_p$$
$$= -4\pi G\{-\phi_0 e^{-\nu_0}\omega[2(\omega\Phi_{1p} - \dot{\Phi}_{2p}) - \omega\phi_0\nu_p] - \phi_0'(2\Phi_{1p}' - \phi_0'\lambda_p) + V_p\}, \tag{23}$$

$$\frac{1}{2}\left[(\nu_p' + \mu_p')\mu_0' + \nu_0'\mu_p' - \lambda_p\left(\frac{1}{2}\mu_0'^2 + \mu_0'\nu_0'\right)\right] - e^{-\nu_0}\ddot{\mu}_p + e^{-\mu_0}\mu_p$$
$$= -4\pi G\{\phi_0 e^{-\nu_0}\omega[2(\omega\Phi_{1p} - \dot{\Phi}_{2p}) - \omega\phi_0\nu_p] + \phi_0'(2\Phi_{1p}' - \phi_0'\lambda_p) + V_p\}, \tag{24}$$

$$\mu_p'' + \nu_p'' + \mu_0'\left(\mu_p' + \frac{1}{2}\nu_p' - \frac{1}{2}\lambda_p'\right) + \nu_0'\left(\frac{1}{2}\mu_p' - \frac{1}{2}\lambda_p' + \nu_p'\right) - \lambda_p\left[\mu_0'' + \nu_0'' + \frac{1}{2}(\mu_0'^2 + \nu_0'^2 + \mu_0'\nu_0')\right] - e^{-\nu_0}(\ddot{\mu}_p + \ddot{\nu}_p)$$
$$= -8\pi G\{\phi_0 e^{-\nu_0}\omega[2(\omega\Phi_{1p} - \dot{\Phi}_{2p}) - \omega\phi_0\nu_p] - \phi_0'(2\Phi_{1p}' - \phi_0'\lambda_p) + V_p\}, \tag{25}$$

where V_p is the perturbation of the potential. Next, from the $\binom{1}{0}$ component of the Einstein equations, we have to linear order

$$2\dot{\mu}_p' - \dot{\lambda}_p\mu_0' + \dot{\mu}_p(\mu_0' - \nu_0') = 16\pi G[\phi_0'\dot{\Phi}_{1p} - \omega\phi_0\Phi_{2p}' + \omega\phi_0'\Phi_{2p}]. \tag{26}$$

In turn, for the components of the scalar field, there are two equations (coming from the real and imaginary parts),

$$\Phi_{1p}'' + \left(\frac{1}{2}\nu_0' + \mu_0'\right)\Phi_{1p}' - e^{-\nu_0}(\ddot{\Phi}_{1p} + 2\omega\dot{\Phi}_{2p}) - \omega^2 e^{-\nu_0}\phi_0\nu_p + \phi_0[\omega^2 e^{-\nu_0} - m^2 + \bar{\lambda}\phi_0^2]\lambda_p$$
$$+ (\omega^2 e^{-\nu_0} - m^2 + 3\bar{\lambda}\phi_0^2)\Phi_{1p} + \frac{1}{2}\phi_0'(\nu_p' - \lambda_p' + 2\mu_p') = 0, \tag{27}$$

$$\Phi_{2p}'' + \left(\frac{1}{2}\nu_0' + \mu_0'\right)\Phi_{2p}' - \frac{1}{2}(\dot{\nu}_p - \dot{\lambda}_p - 2\dot{\mu}_p)\omega e^{-\nu_0}\phi_0 - e^{-\nu_0}(\ddot{\Phi}_{2p} - 2\omega\dot{\Phi}_{1p}) + (-m^2 + \omega^2 e^{-\nu_0} + \bar{\lambda}\phi_0^2)\Phi_{2p} = 0, \tag{28}$$

where $\bar{\lambda}$ is the self-coupling constant from Eq. (11). Note here that combining Eqs. (23)–(27) and letting $\Phi_{2p} = \omega = m = \bar{\lambda} = 0$ one can recover the perturbed equations for the real massless scalar field of Ref. [26] and perform the corresponding calculations for the perturbations.

So far, in this section, we have used the gauge freedom to choose the radial coordinate r such that we may set $\lambda_0 = 0$. We have another gauge freedom left concerning the choice of the metric perturbations ν_p, λ_p, and μ_p. Here, we take the gauge $\nu_p = \lambda_p - 2\mu_p$ [27], which permits us to exclude ν_p and to simplify the equations accordingly. Thus, we are left with six Eqs. (23)–(28) for four functions, λ_p, μ_p, Φ_{1p}, and Φ_{2p}, two of which are first-order equations (constraints).

In solving this system of equations, one can proceed as follows [28,29]. To reduce the number of equations, one can get rid of the function Φ_{2p}. To do this, we exclude the term $\dot{\Phi}_{2p}$ from (27) by expressing $\dot{\Phi}_{2p}$ from any of Eqs. (23)–(25). For example, from Eq. (23), we have

$$\dot{\Phi}_{2p} = \omega\left[\Phi_{1p} - \phi_0\left(\frac{1}{2}\lambda_p - \mu_p\right)\right] - \frac{1}{2\phi_0 e^{-\nu_0}\omega} \times\left\{-\phi_0'(2\Phi_{1p}' - \phi_0'\lambda_p) + V_p + \frac{1}{4\pi G}\left[\mu_p'' + \frac{3}{2}\mu_0'\mu_p' - \frac{1}{2}\mu_0'\lambda_p' - \lambda_p\left(\mu_0'' + \frac{3}{4}\mu_0'^2\right) + e^{-\mu_0}\mu_p\right]\right\}. \quad (29)$$

In turn, taking sums of the components of the Einstein equations (23)+(24) and (23)+(25), one can get rid of $\dot{\Phi}_{2p}$ in these equations as well.

To proceed with the stability analysis, we now assume that the perturbations have the following time dependence,

$$y_p(x^0, r) = \bar{y}_p(r)e^{i\chi x^0}, \quad (30)$$

where y is any of the functions μ, λ, and Φ_1 and the functions $\bar{y}_p(r)$ depend only on the spatial coordinate r. For convenience, we hereafter drop the bar. Then, we get the following set of gravitational equations,

$$\text{Eq.}(23) + \text{Eq.}(24):\ \mu_p'' + \left(\frac{1}{2}\nu_0' + \mu_0'\right)\mu_p' - \left(\mu_0'' + \mu_0'^2 + \frac{1}{2}\mu_0'\nu_0'\right)\lambda_p + (2e^{-\mu_0} + \chi^2 e^{-\nu_0})\mu_p = -8\pi G V_p, \quad (31)$$

$$\text{Eq.}(23) + \text{Eq.}(25):\ \mu_p'' + \lambda_p'' + \left(\mu_0' - \frac{1}{2}\nu_0'\right)(3\mu_p' - \lambda_p') - \left[3\mu_0'' + \nu_0'' + \frac{1}{2}(4\mu_0'^2 + \nu_0'^2 + \mu_0'\nu_0')\right]\lambda_p + (2e^{-\mu_0} + \chi^2 e^{-\nu_0})\mu_p + \chi^2 e^{-\nu_0}\lambda_p = -16\pi G[-\phi_0'(2\Phi_{1p}' - \phi_0'\lambda_p) + V_p], \quad (32)$$

where the perturbation of the potential is $V_p = 2\phi_0(-m^2 + \bar{\lambda}\phi_0^2)\Phi_{1p}$. These two equations are supplemented by the equation for the scalar field (27) with (29). As a result, we obtain the system of three Eqs. (27), (31), and (32) for the three functions μ_p, λ_p, and Φ_{1p}.

For this set of equations, we choose the following boundary conditions at $r = 0$,

$$\lambda_p(0) = \lambda_{p0}, \quad \mu_p(0) = \mu_{p0}, \quad \Phi_{1p}(0) = \Phi_{1p0}, \quad (33)$$

where all functions are even functions.

Using the dimensionless variables (12) and introducing also the dimensionless perturbation $\Phi_{1p} \to \sqrt{4\pi G}\Phi_{1p}$, we can now search for numerical solutions to the set of Eqs. (27), (31), and (32). Together with the boundary conditions (33), this system defines an eigenvalue problem for the frequencies χ^2. The question of stability is thus reduced to a study of the possible values of χ^2. Whenever any values of χ^2 are found to be negative, then the corresponding perturbations will grow, and the configurations in question will be unstable against radial oscillations.

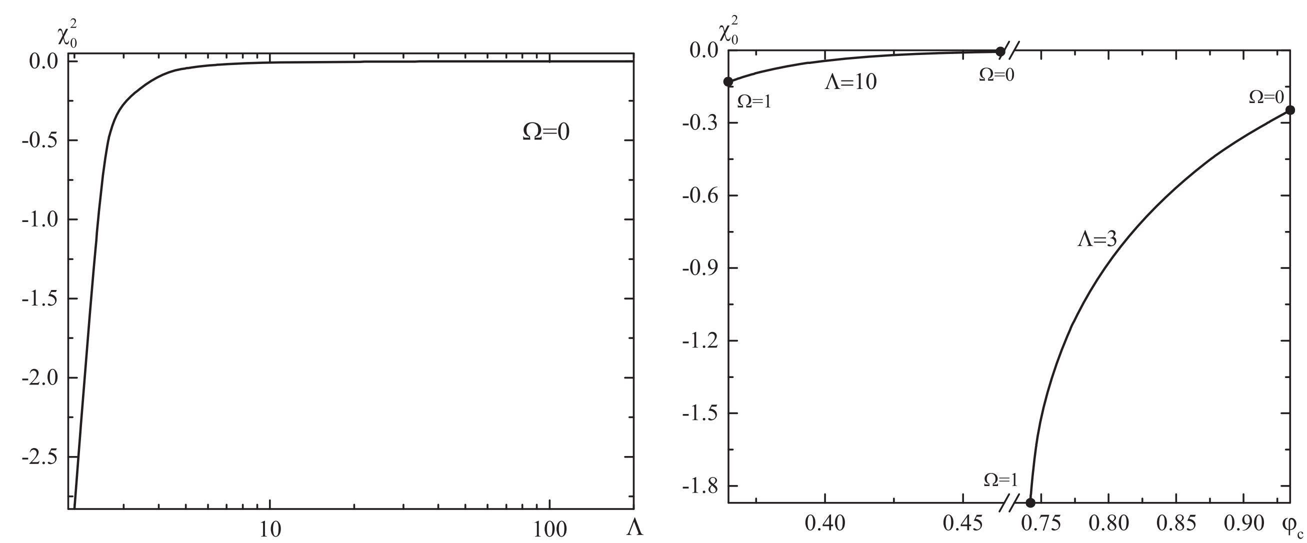

FIG. 5. Left panel: The lowest eigenvalue χ_0^2 is shown as a function of the self-coupling constant Λ for the case of a real scalar field (with $\Omega = 0$ and $\Phi_2 = 0$). For large Λ, $\chi_0^2 \to -0$. Right panel: The lowest eigenvalue χ_0^2 is shown as a function of the central value of the scalar field φ_c for two values of the self-coupling constant Λ in the physically acceptable interval of the boson frequency $0 \leq \Omega \leq 1$.

In performing the stability analysis, we also require that the radial perturbations do not change the total particle number N, given by the expression (22) [1]. This means that the perturbation of the total particle number

$$N_p = 8\pi\omega \int_0^r e^{\mu_0-\nu_0/2}\phi_0^2\left[2\mu_p + \frac{1}{\phi_0}\left(2\Phi_{1_p} - \frac{1}{\omega}\dot{\Phi}_{2p}\right)\right]dr'$$

should be equal to zero. Substituting the perturbed solutions obtained from Eqs. (27), (31), and (32) into this expression, we have to check that the resulting perturbations indeed satisfy the condition $N_p = 0$.

The results of the calculations of the lowest eigenvalue χ_0^2 are shown in Fig. 5. Starting with the case of a real scalar field, i.e., for boson frequency $\Omega = 0$ and $\Phi_2 = 0$, we see in the left panel that in this limiting case the lowest eigenvalue χ_0^2 remains negative for all values of the self-coupling constant Λ. Thus, all corresponding systems are unstable. For the case of a time-dependent scalar field with a finite boson frequency $\Omega \neq 0$, the instability becomes even worse. For any given Λ, as Ω increases, the eigenvalue χ_0^2 decreases, as demonstrated in the right panel of Fig. 5, where, using two values of Λ as examples, the typical dependence of χ_0^2 on the central value of the scalar field φ_c is shown. Thus, the considered topologically nontrivial configurations are always unstable against linear radial perturbations.

IV. CONCLUSION

We have considered equilibrium configurations with a nontrivial wormholelike spacetime topology supported by the complex ghost scalar field with a quartic potential of the Mexican hat form (11). This potential has allowed us to find regular asymptotically flat solutions for an explicitly time-dependent complex scalar field, oscillating with a frequency Ω. We have shown that, depending on the values of Ω and the self-coupling constant Λ, these solutions describe configurations with finite sizes, which may possess a positive or a negative ADM mass, as seen in Fig. 2.

However, the mode stability analysis against linear perturbations has revealed an instability of these systems with respect to radial perturbations possessing reflection symmetry. (Note that in order to demonstrate the instability of the system it is sufficient to consider only this particular subset of perturbations, although one can consider also other types of perturbations, including those given, for example, only on one side of the throat.) We recall that the instability is already present in the limiting case of configurations supported by a real scalar field (with boson frequency $\Omega = 0$), and here we have shown that it is aggravated in the case of an oscillating complex scalar field. In particular, for the complex field, the instability increases with increasing boson frequency Ω, where for any given Λ the square of the lowest eigenfrequency of the radial oscillations becomes increasingly negative (as seen in Fig. 5). Thus, as in the case of real ghost scalar fields considered in the literature before [26,30–34], the use of a complex ghost scalar field does not allow for stable solutions, as demonstrated here for a Mexican hat type potential (11).

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