Hyperimmunity and A-computable universal numberings

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Whether there exists a computable universal numbering for a computable family is the key question in theory of numberings. In a very general setting, this problem was explored in [Yu. L. Ershov, Theory of Numberings, Handbook of Computability Theory, North-Holland; Amsterdam: Stud. Log. Found. Math., Vol. 140, pp. 473-503, 1999]. For sets A that are Turing jumps of the empty set, the problem was treated in [S. A. Badaev, S. S. Goncharov, and A. Sorbi, Computability and Models, 11-44 (2003)] and other papers. In this work, we investigate families of total functions computable relative to hyperimmune and hyperimmune-free oracles.

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Hyperimmunity and *A*-computable universal numberings

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Abstract. Whether there exists a computable universal numbering for a computable family is the key question in theory of numberings. In a very general setting, this problem was explored in [Yu. L. Ershov, *Theory of Numberings*, Handbook of Computability Theory, North-Holland; Amsterdam: Stud. Log. Found. Math., Vol. 140, pp. 473–503, 1999]. For sets A that are Turing jumps of the empty set, the problem was treated in [S. A. Badaev, S. S. Goncharov, and A. Sorbi, Computability and Models, 11–44 (2003)] and other papers. In this work, we investigate families of total functions computable relative to hyperimmune and hyperimmune-free oracles.

Keywords: Universal numbering, Hyperimmune set, Hyperimmune-free degree
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INTRODUCTION

The study of computability leads to a number of very interesting results in mathematics, which determine directions of development in theory of computability. One of these very important directions is the theory of computable numberings.

Computable numberings became more attractive after Kleene, who constructed a universal partial computable function of the class of all partial computable functions, and whose numeration theorem has great meaning in the studying of properties of computable objects. In some sense, we can say that all of these is starting point of the theory of computable numberings.

Any surjective mapping α of the set ω of natural numbers onto a nonempty set S is called a numbering of S. Let α and β be numberings of S. We say that a numbering α is reducible to a numbering β (in symbols, $\alpha \leq \beta$) if there exists a computable function f such that $\alpha(n) = \beta(f(n))$ for any $n \in \omega$. We say that the numberings α and β are equivalent (in symbols, $\alpha \equiv \beta$) if $\alpha \leq \beta$ and $\beta \leq \alpha$.

A numbering $\alpha : \omega \to S$ is called a computable numbering of S in the language L with respect to the interpretation i if there exists a computable function f for which the formula $G(f(n))$ distinguishes an element $\alpha(n)$ in L relative to i, i.e. $\alpha(n) = i(G(f(n)))$ for all $n \in \omega$.

In the case of families F of partial computable functions, for a language L, we take the language of Turing machines. Let $i(M)$ be the function computed by a Turing machine M. In this case, we obtain the standard notion of a computable numbering $\alpha : \omega \to F$ of the family F of partial computable functions: α is computable (relative to i) if and only if there exists a partial computable function $g(n,x)$ such that the functions $\alpha(n)$ and $\lambda x g(n,x)$ coincide for any $n \in \omega$, [3].

Denote by $\deg(\alpha)$ the degree of α, that is, the set $\{\beta | \beta \equiv \alpha\}$ of numberings. The reducibility of numberings is a pre-order relation on the set of all computable numberings of a family S, which we denote by Com(S), and it induces a partial order relation on a set of degrees of the numberings in Com(S), which we also denote by $\leq$. The partially ordered set $\mathscr{R}(\mathrm{S}) = \langle \{\deg(\alpha) | \alpha \in \mathrm{Com(S)}\}, \leq \rangle$ is an upper semilattice, which we call the Rogers semilattice of the family S.

HYPERIMMUNITY

Post proved that simple sets are necessarily m-incomplete and even incomplete for a certain intermediate reducibility called bounded truth-table, Post realized that simple sets could be Turing complete. Indeed, every effectively simple set is T-complete. Thus, Post continued by defining coinfinite c.e. sets with even thinner complements called hypersimple and hyperhypersimple sets. Although these sets also failed to solve Post's problem, they were later shown to have very interesting characterizations which gave considerable information about the structure of nonrecursive c.e. sets

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and about the relationship between an c.e. set and its degree.

Definition 1. *(i) A sequence $\{F_n\}_{n\in\omega}$ of finite sets is a strong (weak) array if there is a recursive function f such that $F_n = D_{f(n)}$ ($F_n = W_{f(n)}$).*

(ii) An infinite set A is hyperimmune (hyper-hyperimmune) if there is no disjoint strong (weak) array $\{F_n\}_{n\in\omega}$ such that $F_n \cap A \neq \emptyset$ for all n.

Definition 2. *(i) A function f majorizes a function g if $f(x) \geq g(x)$ for all x, and f dominates g if $f(x) \geq g(x)$ for almost every x.*

(ii) If $A = \{a_0 < a_1 < a_2 < \cdots$ is an infinite set, the principal function of A is p_A, where $p_A(n) = a_n$.

(iii) A function f majorizes (dominates) an infinite set A if f majorizes (dominates) p_A. Similarly, A majorizes (dominates) f if p_A does.

Theorem 1. *(Kuznecov, Medvedev, Uspenskii) An infinite set A is hyperimmune if and only if no recursive function majorizes A.*

Theorem 2. *(Miller, Martin, [4]) (a) A degree a is hyperimmune if a contains a hyperimmune set, and a is hyperimmune-free otherwise.*

(b) If a is hyperimmune and $a < b$ then b is hyperimmune.

(c) a is hyperimmune iff some function of degree $\leq a$ is majorized by no recursive function.

(d) If $(\exists a)[a < b < a']$, then b is hyperimmune.

(e) Every nonzero degree comparable with $0'$ is hyperimmune.

A-COMPUTABLE UNIVERSAL NUMBERINGS

We consider a family F of total functions, computable relative to an oracle A. A numbering $\alpha : \omega \to F$ is called A-computable if binary function $\alpha(n)(x)$ is A-computable; this concept developed in [5–7]. If A is a computable set, then we deal with a family of computable functions and their classical computable numberings, [1]. Family F is called A-computable if it has an A-computable numbering.

An A-computable numbering α of a family F is universal if any A-computable numbering of F is computably reducible to α. It is well known that, in the classical case, every finite family of c.e. sets has a universal (principal) computable numbering.

S.A. Badaev and S.S. Goncharov in [7] have showed that for every set A such that $\emptyset' \leq_T A$, a finite family S of A-c.e. sets has an A-computable universal numbering if and only if S contains the least set under inclusion. But also they have showed that for an infinite family of A-c.e. sets if $\emptyset' \leq_T A$ then the presence of the least set under inclusion is neither necessary nor sufficient to have an A-computable universal numbering.

Theorem 3. *(Badaev, Goncharov, [7]) For every set A, there is an A-computable family that contains the least set under inclusion but has no A-computable universal numbering.*

Theorem 4. *(Badaev, Goncharov, [7]) For every set A, there is an infinite A-computable family S of sets with pairwise disjoint elements such that S has an A-computable universal numbering.*

We investigate families of total functions computable relative to arbitrary oracles.

Theorem 5. *(Issakhov, [8]) Let F be a family of Σ^0_{n+2}-computable functions. If F contains at least two functions, then F has no universal Σ^0_{n+2}-computable numbering.*

Corollary 6. *Let F be an infinite A-computable family of total functions, where $\emptyset' \leq_T A$. If an A-computable family F contains at least two functions, then F has no A-computable universal numbering.*

Proof of this corollary is obtained by obvious generalizations of previous theorem, which was proved for specific oracle $A = \emptyset^{(n+1)}$, $n \in \omega$.

After this results the following question was arisen: are these results true for oracles $\emptyset < A < \emptyset'$ and oracles which are incomparable with $\emptyset'$? The answers for these questions are the next:

Theorem 7. *Let A be a hyperimmune set. If A-computable family F of total functions contains at least two elements, then F has no universal A-computable numbering.*

Proof. First, note that if A is a hyperimmune set then either $\emptyset <_T A <_T \emptyset'$ or $\emptyset' \leq_T A$.

Let α be an A-computable numbering of the family F consisting at least two functions f and g. Then there exists a number a such that $f(a) \neq g(a)$. We will construct numbering β which satisfies the condition: $\beta \neq \alpha\varphi_e$ for all $e \in \omega$. Construction of β is as follows:

$$\begin{aligned} \beta\langle 0,e,x\rangle &= g, \text{if } \varphi_{e,h(x)}\langle 0,e,x\rangle(a) = f(a); \\ \beta\langle 0,e,x\rangle &= f, \text{ otherwise.} \\ \beta\langle n,e,x\rangle &= \alpha\langle e,x\rangle \text{ for all } n > 0,\ e,x \in \omega, \end{aligned}$$

where $\langle .,.,.\rangle$ is Cantor's function, h is non-majorized A-computable function.

If φ_e is not total then $\beta(x) \neq \alpha\varphi_e(x)$ for some $x \in \omega$. If φ_e is total then the function

$$\hat{h}(x) = min\{s : \varphi_{e,s}\langle 0,e,x\rangle\downarrow\} \tag{1}$$

is computable, and since h is non-majorized there exists a number $b \in \omega$ such that $\hat{h}(b) < h(b)$. Therefore $\varphi_{e,h(b)}\langle 0,e,b\rangle\downarrow$.

If $\alpha\varphi_e\langle 0,e,b\rangle(a) = f(a)$ then by the construction $\beta\langle 0,e,b\rangle = g(a)$. If $\alpha\varphi_e\langle 0,e,b\rangle(a) \neq f(a)$ then by the construction $\beta\langle 0,e,b\rangle = f(a)$. Anyway $\beta\langle 0,e,b\rangle \neq \alpha\varphi_e\langle 0,e,b\rangle$. This completes the proof. □

Theorem 8. *If Turing degree of a set A is hyperimmune-free then every A-computable finite family of total functions has an A-computable universal numbering.*

Proof. Note that degree of a is hyperimmune-free if and only if for any functions g of degree $\leq a$ there exists a total computable function f such that g is majorized by f. Let $F = \{f_0, f_1, ..., f_n\}$ be a given A-computable family of total functions. We fix a collection of sets $A_0, A_1, \cdots, A_n$ satisfying the following conditions for any $i \leq n$:

1. $A_i = \{a_{i0}, a_{i1}, ..., a_{im}\}$, where $m \leq n$;
2. $\forall j \leq n \exists k (j \neq i \Rightarrow f_i(a_{ik}) \neq f_j(a_{ik}))$.

The sets A_i can be defined by the next way: we will compare the functions $\alpha(0), \alpha(1), \cdots, \alpha(k)$ on arguments $\leq k$. Step by step increasing k we will find all necessary elements of A_i.

Let $\Phi_e^A(x,y)$ denote all A-computable binary functions by $e \in \omega$. For corresponding e we can consider $\Phi_e^A(x,y)$ as a numbering of the family F assuming first argument as a number of a function from F and second one as variable of the function and denote it by $\Phi_e^A(x)(y)$. Then we will construct A-computable numbering β of the family F by the following way:

$$\begin{aligned} \beta\langle e,x,s\rangle(y) &= f_i(y), \text{if } \exists i \leq n(f_i(a_{im}) = \Phi_{e,s}^A(x)(a_{im})) \text{ for all } m \leq |A_i|, \\ \beta\langle e,x,s\rangle(y) &= f_0(y), \text{ otherwise.} \end{aligned}$$

From construction it is easy to see that β is an A-computable numbering of the family F. Indeed, β is an A-universal numbering of F. To show it suppose that γ is an A-computable numbering of F. Then there exists $e \in \omega$ such that

$$\gamma(x) = \Phi_e^A(x). \tag{2}$$

If $g(x) = min\{s : \exists i \leq n(f_i(a_{im}) = \Phi_{e,s}^A(x)(a_{im}))\}$ for all $m \leq |A_i|$ then $g(x)$ is A-computable total function. Whereas Turing degree of the set A is hyperimmune-free then there exists a total computable function f such that $g(x) \leq f(x)$ for all $x \in \omega$. It means that

$$\beta\langle e,x,s\rangle(y) = f_i(y) = \Phi_e^A(x)(y) = \gamma(x)(y) \tag{3}$$

for all $y \in \omega$. Indeed, there exists $i \leq n$ such that $f_i(a_{im}) = \Phi_{e,s}^A(x)(a_{im})$ for all $m \leq |A_i|$ but it means that $\beta\langle e,x,s\rangle = f_i$. Since $f_i(a_{im}) = \Phi_{e,s}^A(x)(a_{im})$ for all $m \leq |A_i|$ and $\Phi_e^A(x)(y)$ is a numbering of F, it follows that $f_i(y) = \Phi_e^A(x)(y)$ for all $y \in \omega$. Therefore $\gamma \leq \beta$. This completes the proof. □

Theorem 9. *Let $\emptyset <_T A$. Then any A-computable finite family of total functions has at least two non-equivalent A-computable numberings.*

Proof. Let $F = \{f_0, f_1, ..., f_n\}$ be a given A-computable family of total functions. We construct numberings α and β by the following way:

$$\begin{aligned}
\alpha(i) &= f_i, \text{ if } i \leq k, \\
\alpha(i) &= f_0, \text{ if } i \geq k. \\
\beta(i) &= f_i, \text{ if } i \leq k, \\
\beta(i) &= f_0, \text{ if } i \in A \text{ for } i > k, \text{ and } \beta(i) = f_1, \text{ if } i \notin A \text{ for } i > k.
\end{aligned}$$

Suppose that α and β are equivalent. It follows that there exists computable function g such that $\beta(x) = \alpha g(x)$. Then we have

$$(x > k) \wedge (g(x) = 0 \vee g(x) > k) \Leftrightarrow x \in A \setminus [0,k]. \tag{4}$$

Therefore $A \setminus [0,k]$ is computable set which implies that A is computable too. Contradiction with $\emptyset <_T A$. Theorem 9 is proved. □

Note that from Theorem 9 it follows that there exist non-trivial cases for Rogers semilattices of families in Theorem 8.

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