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Solving of the Regularized Inverse Problem for Elliptic Equations in Cylindrical Coordinates: Analytical Formulas

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Abstract. The continuation inverse problem for a solution to an elliptic equation in cylindrical layer for a model of stationary diffusion process is considered. Cauchy data are given on the outer boundary of the cylindrical layer; need to recover a field at the inner boundary of the cylinder. The problem is reduced to three different Cauchy problems for a second order ordinary differential equation. On the base of necessary minimization conditions of the residual functional analytical formulas for a regularized quasisolution to the inverse problem are derived.

Introduction

General elliptic equation is a universal mathematical model to describe the set of natural and technological processes, including steady-state diffusion of the temperature field. The determination of the temperature at the inner boundary of the cylindrical layer is urgent problem in technological processes, if only the outer boundary is available for observation. A special case of a mathematical model for this problem is the Cauchy problem for the Laplace equation, which is a classic example of ill-posed problem (example Hadamard). Moreover, in the areas of simple form solution of the problem can be obtained in the form of a series [1], but this solution is exponentially unstable with respect to a perturbation of the Cauchy data. This fact allows us to assign this task to a highly uncorrected, according to the classification of inverse problems [2].

Over the past decade there has been significant progress in the solution of inverse problems, namely, the method of quasi-solution [3] and regularization [4] which allows us to solve the task with acceptable accuracy now. In many cases the gradient of the residual functional is expressed in terms of the solution to the adjoint problem [5-7] The method of minimizing the residual functional based on gradient methods is standard method in the numerical solution of the inverse problem. However, analytical methods have an advantage over the numerical methods in accuracy of results and the analysis depending on the parameters of the problem. Analytical formulas for regularized solutions of the initial-boundary value problem for the Laplace equation in a rectangle were obtained for the first time in [8]. The method is based on the system of necessary conditions for a minimum residual functional and construction of the solution of this system in final form. In this paper, we implement this way for the solution of the inverse problem for the three-dimensional steady heat conduction in the medium with cylindrical geometry for the elliptic equation with variable coefficients for second derivatives.

Statement of the Problem

Suppose that the external part of the cylinder with the radius R_2 and height H available for an observer. the cylinder is made from heterogeneous material and the coefficient of thermal conductivity of it depends on the radius r. Finding of the temperature at the inner boundary of the cylinder $r = R_1$ is required, if you know the heat flux and the temperature at the outer boundary $r = R_2$.

We write the mathematical model of the problem. Stationary temperature distribution in a cylindrical layer is described by an elliptic equation of the form:

$$\Lambda u \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(ra(r)\frac{\partial u}{\partial r}\right)+\frac{a(r)}{r^2}\frac{\partial^2 u}{\partial \varphi^2}+a(r)\frac{\partial^2 u}{\partial z^2}=0, \qquad (r,\varphi,z)\in Q \qquad (1)$$

on the next field $Q=\{(r,\varphi,z)\,|\,r\in(R_1,R_2),\varphi\in[0,2\pi),z\in(0,H)\}$

with the initial and boundary conditions:

$$u(r,\varphi,0)=0,\;\; u(r,\varphi,H)=0,\; u(R_2,\varphi,z)=s(\varphi,z), \qquad (2)$$

$$\frac{\partial u}{\partial r}(R_2,\varphi,z)=p(\varphi,z), \qquad (3)$$

and the periodicity conditions for φ:

$$u(r,0,z)=u(r,2\pi,z),\;\; u_\varphi(r,0,z)=u_\varphi(r,2\pi,z), \qquad (4)$$

Need to find a solution on the boundary $r = R_1$:

$$u(R_1,\varphi,z)=q(\varphi,z)-?. \qquad (6)$$

Here, the function $a(r)$ describes the thermal diffusivity of the medium and assumed the known smooth positive function $a(r)$ is strictly bounded away from zero and bounded above.

The problem is solved by quasisolution method, namely, by minimizing the regularized residual functional:

$$J(q)=\frac{R_2}{2}\int_0^H\int_0^{2\pi}(u_r(R_2,\varphi,z;q)-p(\varphi,z))^2d\varphi dz+\frac{R_1}{2}\beta\int_0^H\int_0^{2\pi}q^2(\varphi,z)d\varphi dz\to\min_{q(.,\cdot)}. \qquad (7)$$

It can be shown that the operator of inverse problem is self adjoint, which implies the convexity of the residual functional. Adding the second summand to the functional provides a strongly convexity of it. This implies the existence and uniqueness of the solution of the minimization problem (7) on the convex set of the integrable functions with square of q (z) [9]. Can be also shown, as in [10], what considered functional is differentiable, so the solution of the minimization problem can be found by equating to zero of the Frechet derivative of the functional.

Method of the Solution

We can obtain the condition of minimum residual functional, expressed as a system of equations of the direct and adjoint problems following on the methods of optimization theory, as in [6], [11]:

$$\Lambda u=0,\quad \Lambda v=0,\quad (\mathrm{r},\varphi,\mathrm{z})\in \mathrm{Q} \qquad (8)$$

The boundary conditions have the following form:

$$\begin{aligned}
&u(r,\varphi,0)=0,\ \ u(r,\varphi,H)=0,\\
&v(r,\varphi,0)=0,\ \ v(r,\varphi,H)=0,\\
&v(R_1,\varphi,z)=0,\\
&a(R_2)v(R_2,\varphi,z)-u_r(R_2,\varphi,z)=-p(\varphi,z),\\
&u(R_2,\varphi,z)=0,\ \beta u(R_1\varphi,z)=a(R_1)v_r(R_1,\varphi,z),\\
&v(r,0,z=v(r,2\pi,z),v_\varphi(r,0,z=v_\varphi(r,2\pi,z),\\
&u(r,0,z=u(r,2\pi,z),u_\varphi(r,0,z=u_\varphi(r,2\pi,z),\\
&r\in[R_1,R_2],\varphi\in[0,2\pi],z\in[0,H].
\end{aligned} \tag{9}$$

The Solution of (8) - (9) allow us determine the desired solution of the inverse problem - function $u(r_1,\varphi,z)$. Let have a certain number of harmonics K, L the angle φ and z, and we seek a solution of problem (8) - (9) in the form of series:

$$u(r,\varphi,z)=\sum_1^L\sum_1^K u_{kl}(r)\sin k\varphi\sin\frac{\pi lz}{H},\quad v(r,\varphi,z)=\sum_1^L\sum_1^K r_{kl}(r)\sin k\varphi\sin\frac{\pi lz}{H} \tag{10}$$

For each k, l introduce auxiliary functions U (r), W (r), V (r), which are the solutions of the Cauchy problem

$$\begin{aligned}
&\frac{1}{r}\left(ra(r)U'\right)'-a(r)\left(\frac{k^2}{r^2}+\frac{\pi^2l^2}{H^2}\right)U=0,\ r\in[R_1,R_2]\\
&U'(R_1)=1,\ U(R_2)=0,
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\frac{1}{r}\left(ra(r)W'\right)'-a(r)\left(\frac{k^2}{r^2}+\frac{\pi^2l^2}{H^2}\right)W=0,\quad \mathrm{r}\in r\in[R_1,R_2]\\
&W'(R_2)=0,\ \ W(R_2)=1
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&\frac{1}{r}\left(a(r)rV'\right)'-a(r)\left(\frac{k^2}{r^2}+\frac{\pi^2l^2}{H^2}\right)V=0,\ \mathrm{r}\in r\in[R_1,R_2]\\
&V'(R_1)=1,V(R_2)=0.
\end{aligned} \tag{13}$$

The solution of the above problems can be obtained by the Runge-Kutta methods, either analytically, based on the apparatus of the Bessel functions if the thermal conductivity is constant. Equations for the functions U, V, W are identical, they are reduced to the formation:

$$\frac{d^2R}{dr^2}+\frac{(p(r)+1)}{r}\frac{dR}{dr}-\left(\frac{k^2}{r^2}+\frac{\pi^2l^2}{H^2}\right)R=0,\ \ p(r)=\ln(a(r)). \tag{15}$$

Expanded in a Fourier series data Cauchy (2) - (3), calculate the coefficient of double series:

$$p(\varphi,z)=\sum_1^K\sum_1^L p_{kl}\sin k\varphi\sin\frac{\pi lz}{H},\qquad s(\varphi,z)=\sum_1^K\sum_1^L s_{kl}\sin k\varphi\sin\frac{\pi lz}{H}.$$

After substituting the solution in the form (10) and the boundary conditions in the problem (8) - (9) we obtain a system of two-point boundary value problems for functions $u_{kl}(r)$ and $v_{kl}(r)$:

$$\frac{1}{r}\left(a(r)ru'_{kl}\right)'-a(r)\left(\frac{k^2}{r^2}+\frac{\pi^2 l^2}{H^2}\right)u_{kl}=0,$$
$$\frac{1}{r}\left(a(r)rv'_{kl}\right)'-a(r)\left(\frac{k^2}{r^2}+\frac{\pi^2 l^2}{H^2}\right)v_{kl}=0, \tag{16}$$

$$v_{kl}(R_1)=0,$$
$$a(R_2)v_{kl}(R_2)-u'_{kl}(R_2)=-p_{kl},$$
$$u_{kl}(R_2)=s_{kl},$$
$$\beta u_{kl}(R_1)-a(R_2)v'_{kl}(R_1)=0. \tag{17}$$

We seek a solution of problem (16) - (17) for each pair (k, l) as a linear combination of solutions of auxiliary problems (11) - (13):

$$u_{kl}(r)=AU(r)+CW(r),\ v_{kl}(r)=BV(r) \tag{18}$$

We substitute the expression (18) in terms of (17) for each pair of values (k, l), from which we obtain a system of linear equations for the unknown coefficients A, B, C:

$$a_2BV(R_2)-A=-p_{kl},\ CW(R_2)=s_{kl},\ \beta AU(R_1)+\beta CW(R_1)-a_2B=0.$$

Hence it follows that:

$$C=\frac{s_{kl}}{W(R_2)},\quad \mathrm{A}=a_2BV(R_2)+p_{kl},\quad B=\frac{-\beta U(R_1)p_{kl}-\beta s_{kl}W(R_1)/W(R_2)}{a_2(\beta V(R_2)U(R_1)-1)},$$
$$A=\frac{-\beta^2 s_{kl}V(R_2)W(R_1)/W(R_2)+p_{kl}}{\beta V(R_2)U(R_1)-1}.$$

We write the partial sum of the series (10) at the point R1, thereby determining the desired temperature at the inner boundary of the cylinder:

$$q(\varphi,z)=u(R_1,\varphi,z)=\sum_1^L\sum_1^K u_{kl}(R_1)\sin k\varphi\sin\frac{\pi l z}{H},$$

where

$$u_{kl}(R_1)=\frac{-\beta^2 s_{kl}V(R_2)W(R_1)/W(R_2)+p_{kl}}{\beta V(R_2)U(R_1)-1}U(R_1)+\frac{s_{kl}}{W(R_2)}W(R_1).$$

Summary

Numerical implementation of the method we have described above will be the subject of a separate publication. Here, as an illustration, we show in Figure 1, the results of calculating the temperature on the surface of a cylinder of radius $R_1=1$, $R_2=2$, height H.

We have implemented the above algorithm for solving the inverse problem for the environment with a constant coefficient of thermal conductivity. In Figure 1 we compare the result of the restoration with the exact solution. On the result of the calculation is influenced by both the parameters of the problem and the regularization parameters β and the number of harmonics accounted L.

Preliminary calculations show that our method allows to accurately solve the Cauchy problem for an elliptic equation in a cylindrical layer for smooth data of the problem. Note that the method does not allow a good accuracy restore discontinuous functions.

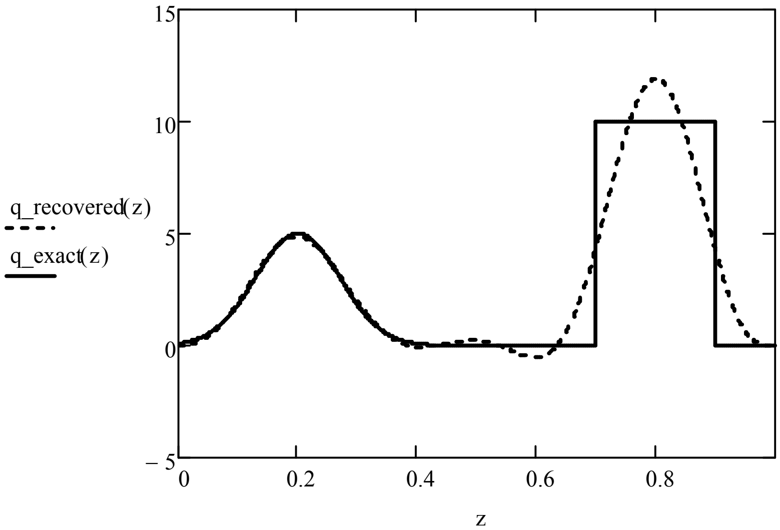

Fig 1 - The result of the inverse problem solution for parameters calculation: $R_1 = 1$, $R_2 = 2$, the number of harmonics of $L = 10$, $\beta = 0$.

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