Experimental Study of Gluing Bifurcations

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***Abstract*— We report on the experimental investigation of gluing bifurcations in the electronic circuit.**

I. Introduction

For most of scenarios of transition from order to chaos in low-dimensional dynamical systems theoretical investigations were followed by accurate experimental verification which not only recovered the qualitative picture of the transition but were also able to resolve its quantitative details. Universal sequence of period-doubling bifurcations, as well as transitions to chaos via the breakup of the quasiperiodic state and via different kinds of intermittency have been objects of numerous mechanical, hydrodynamical, optical, chemical etc. experiments (see references e.g. in [1], [2]).

A scenario which, albeit well understood theoretically, has received less attention from the experimentalists, is the route to chaos via the sequence of the so-called *gluing bifurcations*. In the course of these bifurcations, the stable periodic orbits come together, recombine and form the new periodic orbits which are more complicated than the original ones. Recombination occurs via the formation in the phase space of the system of two orbits, homoclinic to a saddle point. Homoclinic bifurcations are well-known in chaotic dynamics: in the famous Lorenz equations [3], the birth of the chaotic set in the phase space occurs via the so-called "homoclinic explosion" [4], [5]. The main difference between the homoclinic explosion and the gluing bifurcation lies in the number of newborn periodic orbits as well as in their stability. The explosion creates simultaneously the countable set of unstable periodic orbits which form the kind of "skeleton" of the emerging chaotic attractor. In contrast, a gluing bifurcation, taken alone, produces just one or two stable periodic orbits; however, in the course of the sequence of such bifurcations the periodic orbits get more and more complicated, until the whole development culminates in the birth of the chaotic attractor.

Unlike limit cycles, homoclinic trajectories in generic dynamical systems are structurally unstable. Furthermore, the accomplishment of the gluing scenario requires either the perfect mirror symmetry of the system, or the ability to track in the parameter space the sequences of codimension-two phenomena. For this reason, the gluing bifurcations pose a difficult object for experimental studies: they are sensitive to fluctuations, as well as to imperfections of the experimental setup. Most occurrences of gluing bifurcations were reported in theoretical and numerical studies in the context of hydrodynamics [6]–[8], nematic liquid crystals [9], [10] and optothermal nonlinear devices [11]. In the experiments, separate gluing bifurcations were detected in the Taylor-Couette flow of the viscous fluid between two concentric cylinders [12]. Here, we report on our experimental investigation of the sequence of gluing bifurcations in the electronic circuit.

II. Theoretical predictions

A. General theory

The main ingredient of the gluing bifurcation in the continuous dissipative dynamical system is the equilibrium point of the saddle type with one-dimensional unstable manifold. We restrict ourselves to the situation when two components of this manifold are symmetric (like in the case of the Lorenz equations). When the parameters of the system are varied, the unstable manifold of the saddle can return back along its stable manifold: the homoclinic orbit is formed. Symmetry ensures that every such orbit possesses a twin. Phenomena which accompany the birth/destruction of homoclinic orbits, depend on the leading eigenvalues of the Jacobian of the equations at the saddle point. In the considered situation, there is just one positive eigenvalue, denoted below as λ_+. Let the closest to zero negative eigenvalue λ_- be real. Stability of the bifurcating solutions depends on the "saddle index" $\nu = |\lambda_-|\lambda_+$ that indicates which of the two properties – contraction or expansion – dominates the phase space in the neighborhood of the fixed point. In the absence of symmetry, destruction of the single homoclinic trajectory creates the unique periodic orbit which is stable if $\nu > 1$ and unstable otherwise. Presence of the second, symmetric homoclinic trajectory enriches the dynamics: in this case, under $\nu < 1$ the countable set of unstable periodic orbits, as well as a continuum of recurrent trajectories is born from the pair of homoclinic orbits [5]; this is a decisive step in the subsequent formation of the Lorenz attractor. The situation for $\nu > 1$ is less complicated: here two stable periodic orbits approach the saddle point, swell into the homoclinic orbits and are "glued together". When the pair of homoclinic trajectories breaks up, the new stable symmetric periodic orbit is left in the phase space: it is obtained by concatenation of the previously existing ones. The number of loops (turns) of the trajectory in the phase space is doubled, like in the case of the period-doubling bifurcation. Notably, the temporal period gets unbounded near the bifurcation and becomes infinite when the homoclinic trajectories are formed.

Further variation of parameter can result in the sequence of secondary gluing bifurcations. Before this can happen, the symmetry-breaking should take place: the newborn symmetric periodic orbit loses stability in the course of the pitchfork bifurcation, and two mutually symmetric orbits bifurcate from it. These two orbits approach the unstable manifold of the saddle and coalesce in the next gluing bifurcation. As a result, the new stable periodic orbit is born, which has four times more loops than the original ones. The subsequent scenario consists of the alternating gluing- and symmetry-breaking bifurcations which eventually end in the formation of the chaotic attractor which has a two-lobe shape, similar to that of the Lorenz attractor. It has been demonstrated [6], [13] that the sequence of the bifurcational values of the parameter converges at the exponential rate; in contrast to the period-doubling scenario, this rate is not the unique universal constant but is predetermined by the value of the saddle-index $\nu > 1$.

Remarkably, the attractor which is formed in the course of this bifurcation scenario, occupies a certain intermediate position between order and chaos: the Fourier spectrum of the trajectory is neither discrete (like in case of the regular dynamics) nor absolutely continuous (like in case of a chaotic or stochastic process) but is supported by the fractal set [14].

In the absence of symmetry between the components of the unstable manifold, the gluing bifurcation becomes a codimension-two phenomenon; on the plane of two parameters, there are numerous paths which lead from order to chaos via the formation of homoclinic orbits; each of these paths is characterized by its own scaling constants [15]–[17].

B. The model set of equations

For experimental modeling by the appropriate electronic circuit, we seek a dynamical system of the minimal possible order and with simple algebraic form. Gluing bifurcations can occur in the three-dimensional phase space, like that of the Lorenz equations. However, numerical evidence indicates that in the parameter space of the "canonical" Lorenz system the homoclinic bifurcations can occur only for $\nu < 1$. Therefore, we introduce a modification and consider the set of equations

$$\begin{aligned} \dot{x} &= \sigma(y - x) + A\,y\,z \\ \dot{y} &= R\,x - y - x\,z \\ \dot{z} &= x\,y - b\,z \end{aligned} \qquad (1)$$

Here, σ, R and b are the conventional Lorenz parameters, whereas A parameterizes the additional nonlinear term in the first equation. At A=0 Eqs (2) turn into the Lorenz equations. In fact, the system (2) is the re-parameterization of the model, employed in [6] for the study of thermal convection in the layer of fluid subjected to high-frequency modulation of gravity. We fix the traditional values of the parameters σ=10 and b=8/3 [3], and vary the remaining parameters R and A.

Equations (2) are invariant with respect to the transformation $\{x \to -x,\ y \to -y\}$. The origin $x = y = z = 0$ is the equilibrium which is stable for $R < 1$ and is a saddle with one-dimensional unstable manifold in the parameter range $R > 1$. Like in the original Lorenz equations, variation of parameters can produce pairs of structurally unstable homoclinic orbits to this equilibrium; these orbits leave the vicinity of the origin along the xy-plane and return to it along the z-axis. The saddle index equals

$$\nu = \frac{2\,b}{-\sigma - 1 + \sqrt{(\sigma-1)^2 + 4\sigma\,R}}. \qquad (2)$$

Notably, the value of ν is A-independent, which allows to study the effects of additional nonlinearity under constant saddle index. It is straightforward to see that ν is smaller than 1 for $R > R_\nu = (b + \sigma)(b + 1)/\sigma$ and exceeds 1 otherwise. Numerical integration indicates that increase of A lowers the critical value of R, required for the formation of the pair of homoclinic orbits. Therefore, at small positive values of A the homoclinic explosion and the Lorenz scenario of transition to chaos are observed, whereas the sufficiently large values of A ensure the inequality $\nu > 1$ and enable thereby the sequence of gluing bifurcations.

III. Experimental setup

Our electronic circuit is a modification of the circuit which was employed in [18] to mimic the dynamics on the Lorenz attractor. As seen in Fig. 1, the nonlinear term $\sim yz$ in the first equation of (2) is modeled by the additional analog multiplier.

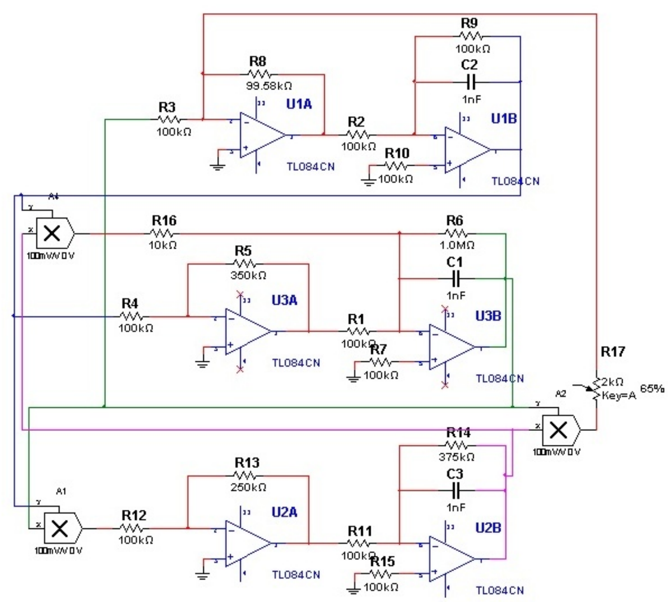

Fig. 1. Electronic circuit for modeling Eqs (2). The values of the resistors and capacitors are shown on the scheme. The analog multipliers (denoted by crosses) are of the type AD633AN.

The cited values of electric characteristics correspond to fixed values $\sigma = 10$ and $b = 8/3$ in Eqs (2). Variation of parameters A and R is implemented through variation of respective resistances $R17$ and $R5$; the value of R equals 0.01 of the resistance $R5$ in $k\Omega$.

To ensure that all voltages in the circuit are inside the operating range of dynamical multipliers (from -10 to 10 V), we rescale the original dependent and independent variables: we assume that the dimensionless variables x, y and z are measured in volts, and proceed to $u = x/5$, $v = y/5$, $w = z/10$. For the time measured in "seconds", we introduce $\tau = t/T$ with T=100. This recasts Eqs (2) into

$$\begin{aligned} u' &= T\,(\sigma(v-u) + 10\,A\,v\,w) \\ v' &= T\,(R\,u - v - 10\,u\,w) \\ z' &= T\,(2.5\,u\,v - b\,w) \end{aligned} \tag{3}$$

where prime denotes differentiation with respect to time τ.

IV. Results

We performed measurements for different values of the parameters A and R. The equilibrium u=v=w=0 is a saddle-point in the range $R > 1$. On the parameter plane of R and A the pair of principal homoclinic orbits to this equilibrium exists on the curve which passes through the point $A = 0, R = 13.926\ldots$ [4]. For low values of A this curve corresponds to the homoclinic explosion of the Lorenz type. According to the theory, explosion should be replaced by the gluing bifurcation when the corresponding value of R gets below R_ν=209/45=4.6444. . .; this corresponds to the range $A \geq 4$. Indeed, we observed in the experiment the Lorenz-like chaotic attractors for $A < 4$ and gluing bifurcations for $A > 4$.

We illustrate the bifurcation scenario with the plots which show the transformation of projections of the phase portraits which correspond to increase of A at constant values of other parameters.

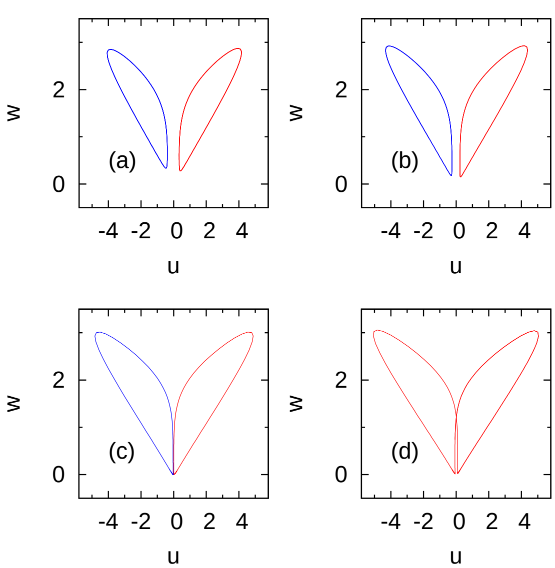

Fig. 2. Principal gluing bifurcation: Evolution of the attracting set. (a,b): two stable periodic orbits. (c): formation of two trajectories, homoclinic to the equilibrium. (d): stable self-symmetric orbit with two loops.

By varying initial conditions, we are able to identify in the phase space of the circuit two attracting closed trajectories which are almost symmetric to each other (Fig.2a). As the parameter A is increased, these two orbits come closer (Fig.2b), approach the invariant manifolds of the saddle equilibrium at the origin and form the pair of symmetric homoclinic orbits to this saddle (Fig.2c). As soon as the parameter A gets another increment, the homoclinic orbits break up and disappear: the only attractor of the system is shown in Fig.2d: it is the periodic orbit with two loops which is invariant under the symmetry transformation $\{u \to -u,\, v \to -v\}$.

According to our measurements, close to the homoclinic bifurcation the period T_0 of the oscillations reproduces the known logarithmic asymptotics: $T_0(A) \sim \log\big(1/(A_{\rm hom}-A)\big)$. This is visualized in Fig.3.

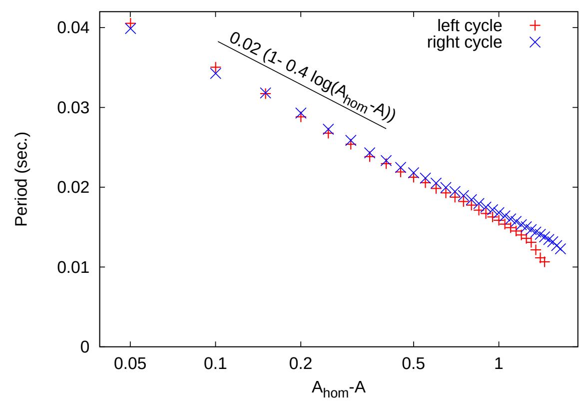

Fig. 3. Growth of the period of oscillations close to the homoclinic bifurcation.

As seen in Fig.4, the oscillations are strongly anharmonic: the overwhelming part of the period is spent in nearly motionless state. This corresponds to long hovering of the trajectory in the vicinity of the saddle point.

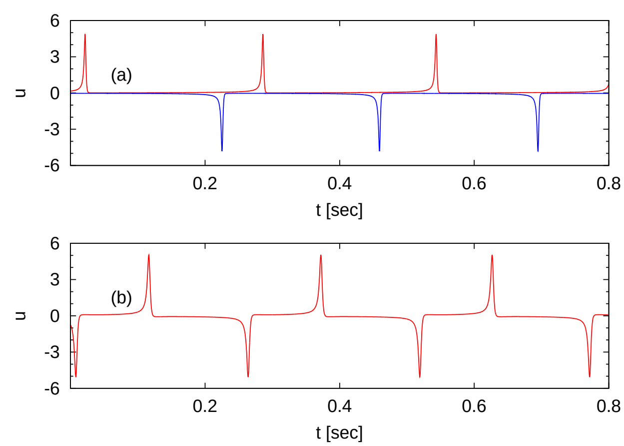

Fig. 4. Temporal evolution close to the gluing bifurcation.
(a): oscillations on limit cycles with 1 loop prior to the bifurcation;
(b): oscillations on the limit cycle with 2 loops.

Further stages of the evolution of the attracting trajectory are sketched in Fig.5. The symmetric limit cycle with two turns (Fig.5a) loses stability as a result of the pitchfork bifurcation. One of the two resulting stable asymmetric limit cycles is shown in Fig.5b. As the parameter is further increased, these two limit cycles approach the equilibrium, coalesce, and the secondary gluing bifurcation takes place: two orbits with 2 turns produce a single stable orbit with 4 turns (Fig.5c). The resolution of our measurements as well as the inevitable asymmetry in the circuit did not yet allow us to resolve the further stages of the gluing process; the bifurcation sequence rapidly converges, and the chaotic attractor (Fig.5d) is established.

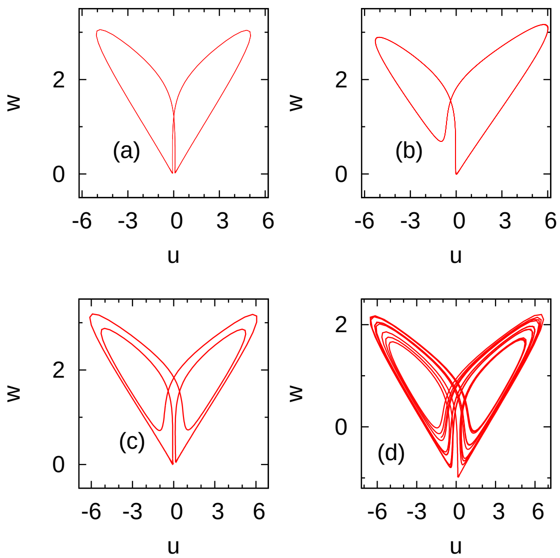

Fig. 5. Further volution of the attracting set.
(a): stable symmetric orbit with two turns; (b): asymmetric orbit with two turns; (c): symmetric orbit with 4 turns born from the secondary gluing bifurcation; (d): chaotic attractor.

V. Conclusion

We have demonstrated in the experiment on the electronic circuit the first stages of the sequence of gluing bifurcation. Experimental results match well the theoretical predictions. In further studies we intend to refine the measurements, resolve the scaling properties in physical characteristics of oscillations as well as in the parameter space, and investigate the properties of the eventual chaotic attractors.

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