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On the Stability of Circular Orbits of a Test Body in the Restricted Three-Body Problem in GR Mechanics

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Abstract—The paper considers the problem of orbital stability of a circular motion of a test body in the restricted three-body problem, where perturbations from the second body are of the order of relativistic corrections to the motion of a test body from the central body. All bodies do not possess proper rotation.

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1. INTRODUCTION

The three-body problem is of deep theoretical and extensive practical importance in modern classical celestial mechanics, but a comprehensive and detailed consideration in the framework of relativistic celestial mechanics is impeded by the lack of relevant observational data and special experiments, as well as the fact that all analytical and theoretical calculations face enormous technical difficulties. The lack of observational data is due mainly to the presence of many classical factors that sometimes far exceed the general-relativistic effects, and to distinguish them from the general background is in most cases not possible. In the works dedicated to the relativistic three-body problem, some peculiarities of particular problems were considered [1–3]. But in some cases, as we believe, cosmogonic general-relativistic effects can be substantial, even determining the evolution of a system of bodies, especially when it concerns stable orbital motions of celestial bodies. Therefore, the development of relativistic planetary cosmogony and the study of the evolution of planetary systems in the framework of general relativity is very important.

2. EQUATION OF MOTION

Previously we have considered the orbital stability in the problem of two rotating bodies [4, 5]. Here we consider the problem of orbital stability of circular motion of a test body in the restricted three-body problem, where disturbances from a circular orbital motion of the second body in the field of motion of the test body (in the plane of motion of the second body) are of order of relativistic corrections to the motion of a test body from the central body,

$$U_1 \ll c^2, \quad U_2 \ll U_1, \quad U_1/U_2 \approx v^2/c^2, \tag{1}$$

where U_1, U_2 are the potentials of the central and second bodies, respectively. All the bodies do not have a proper rotation. The resting position of the central body coincides with the origin of coordinates, the second body moves along a circle around the central body (the first one) and is not subject to disturbance. The test body moves in a perturbed circular orbit. This problem belongs to the class of quasi-Keplerian ones, and we apply to it the adiabatic theory of motion, well established in GR mechanics, developed by M.M. Abdildin [4]. The relativistic Lagrange function of the problem has the form [4–6]

$$\begin{aligned} L = {} & \frac{m_2 v_2^2}{2} + \frac{m_3 v_3^2}{2} + \gamma\frac{m_1 m_2}{|\mathbf{r}_2|} + \gamma\frac{m_1 m_3}{|\mathbf{r}_3|} \\ & + \gamma\frac{m_2 m_3}{|\mathbf{r}_2 - \mathbf{r}_3|} + \frac{1}{8c^2}(m_2 v_2^4 + m_3 v_3^4) \\ & + \frac{\gamma}{2c^2}\left[\frac{3m_1 m_2}{|\mathbf{r}_2|}v_2^2 + \frac{3m_1 m_3}{|\mathbf{r}_3|}v_3^2 + \frac{m_2 m_3}{|\mathbf{r}_2 - \mathbf{r}_3|}\right. \\ & \left.\times\left(3v_2^2 + 3v_3^2 - 7(\mathbf{v}_2\mathbf{v}_3)\right)\right] - \frac{\gamma}{2c^2} \\ & \times\frac{(\mathbf{v}_2(\mathbf{r}_2 - \mathbf{r}_3))(\mathbf{v}_3(\mathbf{r}_2 - \mathbf{r}_3))}{|\mathbf{r}_2 - \mathbf{r}_3|^2} - \frac{\gamma^2}{c^2}m_1 m_2 m_3 \\ & \times\left(\frac{1}{|\mathbf{r}_2||\mathbf{r}_3|} + \frac{1}{|\mathbf{r}_2||\mathbf{r}_2 - \mathbf{r}_3|} + \frac{1}{|\mathbf{r}_3||\mathbf{r}_2 - \mathbf{r}_3|}\right) \\ & - \frac{\gamma^2}{c^2}m_1 m_2 m_3\left(\frac{1}{|\mathbf{r}_2||\mathbf{r}_3|} + \frac{1}{|\mathbf{r}_2||\mathbf{r}_2 - \mathbf{r}_3|}\right) \\ & - \frac{\gamma^2}{c^2}m_1 m_2 m_3\frac{1}{|\mathbf{r}_3||\mathbf{r}_2 - \mathbf{r}_3|}. \end{aligned} \tag{2}$$

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Our task is to find the evolution equation of motion for the test (third) body which describes the average change of its orbital momentum. To do that, we write down the orbital angular momentum of the test body,

$$\mathbf{M} = [\mathbf{r}_3, \mathbf{p}_3], \tag{3}$$

and its time derivative

$$\dot{\mathbf{M}} = [\dot{\mathbf{r}}_3, \mathbf{p}_3] + [\mathbf{r}_3, \dot{\mathbf{p}}_3]\,, \tag{4}$$

where

$$\dot{\mathbf{r}}_3 = \frac{\partial H}{\partial \mathbf{p}_3}, \qquad \dot{\mathbf{p}}_3 = -\frac{\partial H}{\partial \mathbf{r}_3} \tag{5}$$

are found by means of the Hamilton equations from the Hamilton function of the system,

$$H = \mathbf{v}_i \frac{\partial L}{\partial \mathbf{v}_i} - L. \tag{6}$$

The explicit form of the Hamilton function is

$$\begin{aligned} H &= \frac{p_3^2}{2m_3} + \frac{p_2^2}{2m_2} - \gamma\bigg(\frac{m_1 m_2}{|\mathbf{r}_2|} + \frac{m_1 m_3}{|\mathbf{r}_3|} \\ &+ \frac{m_2 m_3}{|\mathbf{r}_2 - \mathbf{r}_3|}\bigg) - \frac{1}{8c^2}\left(\frac{p_3^4}{2m_3^3} + \frac{p_2^4}{2m_2^3}\right) \\ &+ \frac{\gamma}{2c^2|\mathbf{r}_2 - \mathbf{r}_3|}\bigg(7(\mathbf{p}_3\mathbf{p}_2) + \frac{\mathbf{p}_3(\mathbf{r}_2 - \mathbf{r})}{|\mathbf{r}_2 - \mathbf{r}_3|^2} \\ &\times \frac{\mathbf{p}_2(\mathbf{r}_2 - \mathbf{r})}{|\mathbf{r}_2 - \mathbf{r}_3|^2}\bigg) - \frac{3\gamma}{2c^2}\left(\frac{m_3}{|\mathbf{r}_2 - \mathbf{r}_3|} + \frac{m_1}{|\mathbf{r}_2|}\right)\frac{p_2^2}{m_2} \\ &- \frac{3\gamma}{2c^2}\left(\frac{m_2}{|\mathbf{r}_3 - \mathbf{r}_2|} + \frac{m_1}{|\mathbf{r}_3|}\right)\frac{p_3^2}{m_3} \\ &+ \frac{\gamma^2}{c^2} m_1 m_2 m_3 \bigg(\frac{1}{|\mathbf{r}_2||\mathbf{r}_3|} + \frac{1}{|\mathbf{r}_2||\mathbf{r}_2 - \mathbf{r}_3|} \\ &+ \frac{1}{|\mathbf{r}_3||\mathbf{r}_2 - \mathbf{r}_3|})\bigg) + \frac{\gamma^2}{2c^2}\bigg(\frac{m_1 m_2(m_1 + m_2)}{|\mathbf{r}_2|^2} \\ &+ \frac{m_1 m_3(m_1 + m_3)}{|\mathbf{r}_3|^2} + \frac{m_2 m_3(m_2 + m_3)}{|\mathbf{r}_2 - \mathbf{r}_3|^2}\bigg), \end{aligned} \tag{7}$$

and the derivatives of the canonical variables of the test body are

$$\begin{aligned} \dot{\mathbf{r}}_3 &= \frac{\mathbf{p}_3}{m_3}\left[1 - \frac{1}{c^2}\left(\frac{p_3^2}{m_3^2} + 3\gamma\left(\frac{m_2}{|\mathbf{r}_2 - \mathbf{r}_3|} + \frac{m_1}{|\mathbf{r}_3|}\right)\right)\right] \\ &+ \frac{\gamma}{2c^2|\mathbf{r}_2 - \mathbf{r}_3|}\left(7\mathbf{p}_2 + \frac{(\mathbf{r}_2 - \mathbf{r}_3)(\mathbf{p}_2(\mathbf{r}_2 - \mathbf{r}_3))}{|\mathbf{r}_2 - \mathbf{r}_3|^2}\right), \end{aligned} \tag{8}$$

$$\begin{aligned} \dot{\mathbf{p}}_3 &= -\gamma\frac{m_1 m_3}{|\mathbf{r}_3|^3}\mathbf{r}_3 + \gamma\frac{m_2 m_3\,(\mathbf{r}_2 - \mathbf{r}_3)}{|\mathbf{r}_2 - \mathbf{r}_3|^3} \\ &- \frac{7\gamma}{2c^2}\frac{(\mathbf{p}_2\mathbf{p}_3)(\mathbf{r}_2 - \mathbf{r}_3)}{|\mathbf{r}_2 - \mathbf{r}_3|^3} - \frac{\gamma}{2c^2|\mathbf{r}_2 - \mathbf{r}_3|^3} \\ &\times \bigg[\frac{3(\mathbf{p}_3(\mathbf{r}_2 - \mathbf{r}_3))(\mathbf{p}_2(\mathbf{r}_2 - \mathbf{r}_3))(\mathbf{r}_2 - \mathbf{r}_3)}{|\mathbf{r}_2 - \mathbf{r}_3|^2} \\ &- \Big(\mathbf{p}_2(\mathbf{p}_3(\mathbf{r}_2 - \mathbf{r}_3))) + \mathbf{p}_3(\mathbf{p}_2(\mathbf{r}_2 - \mathbf{r}_3))\Big)\bigg] \\ &+ \frac{3\gamma}{2c^2}\bigg[\frac{\mathbf{p}_2^2}{m_2}\frac{m_3(\mathbf{r}_2 - \mathbf{r}_3)}{|\mathbf{r}_2 - \mathbf{r}_3|^3} - \frac{\mathbf{p}_3^2}{m_3}\bigg(\frac{m_2(\mathbf{r}_2 - \mathbf{r}_3)}{|\mathbf{r}_2 - \mathbf{r}_3|^3} \\ &+ \frac{m_1\mathbf{r}_3}{|\mathbf{r}_3|^3}\bigg)\bigg] + \frac{\gamma^2 m_1 m_2 m_3}{c^2}\bigg[\frac{\mathbf{r}_3}{|\mathbf{r}_2||\mathbf{r}_3|^3} \\ &+ \frac{\mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3||\mathbf{r}_3|^3} - \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3|\mathbf{r}_3|}\bigg] \\ &+ \frac{\gamma^2}{c^2}\bigg[\frac{m_1 m_3(m_1 + m_3)\mathbf{r}_3}{|\mathbf{r}_3|^4} \\ &- \frac{m_2 m_3(m_2 + m_3)(\mathbf{r}_2 - \mathbf{r}_3)}{|\mathbf{r}_2 - \mathbf{r}_3|^4}\bigg]. \end{aligned} \tag{9}$$

Substituting these expressions into (4), we obtain the equation for the angular momentum:

$$\begin{aligned} \dot{\mathbf{M}} &= \bigg[\gamma m_2 m_3 - \frac{7\gamma}{2c^2}(\mathbf{p}_2\mathbf{p}_3) - \frac{3\gamma}{2c^2|\mathbf{r}_2 - \mathbf{r}_3|^2} \\ &\times(\mathbf{p}_2(\mathbf{r}_2 - \mathbf{r}_3))(\mathbf{p}_3(\mathbf{r}_2 - \mathbf{r}_3)) + \frac{3\gamma m_2 m_3}{2c^2} \\ &\times\left(\frac{\mathbf{p}_2^2}{m_2^2} + \frac{\mathbf{p}_3^2}{m_3^2}\right) - \frac{\gamma^2}{c^2}m_1 m_2 m_3\left(\frac{1}{|\mathbf{r}_2|} + \frac{1}{|\mathbf{r}_3|}\right)\bigg] \\ &\times\frac{[\mathbf{r}_3, \mathbf{r}_2]}{|\mathbf{r}_2 - \mathbf{r}_3|^3} - \frac{\gamma}{2c^2|\mathbf{r}_2 - \mathbf{r}_3|^3}\Big((\mathbf{p}_3(\mathbf{r}_2 - \mathbf{r}_3))[\mathbf{r}_3, \mathbf{p}_2] \\ &-(\mathbf{p}_2(\mathbf{r}_2 - \mathbf{r}_3))[\mathbf{r}_2, \mathbf{p}_3] + 2(\mathbf{p}_2(\mathbf{r}_2 - \mathbf{r}_3))\mathbf{M}\Big) \\ &+\frac{7\gamma}{2c^2|\mathbf{r}_2 - \mathbf{r}_3|}[\mathbf{p}_2, \mathbf{p}_3]. \end{aligned} \tag{10}$$

3. AVERAGING THE EQUATION OF MOTION

To obtain the evolutionary equations of motion, one needs to integrate Eq. (10) for the repetition period of the system configurations T (the synodic period of the test body):

$$\overline{\dot{\mathbf{M}}} = \frac{1}{T}\int_0^T \dot{\mathbf{M}}dt, \tag{11}$$

where

$$T = \frac{2\pi}{\omega_2 - \omega_3}. \tag{12}$$

The perturbed motion of the test body is described by the expression

$$r = r_{\text{kep}} + r_p + r_{\text{rel}}, \tag{13}$$

where

$$r_{\rm kep} = \frac{p}{1 + e\cos\omega_3 t} \qquad (14)$$

describes the unperturbed motion, and in our case $e = 0$; r_p describes the classical perturbation from the second body, and the third term is the relativistic correction [7]. A classic consideration of this problem has shown [8] that the perturbation from the second body in the evolutionary equation vanishes. Noting that in our consideration the order of the classical perturbation corresponds to the relativistic one, see (1), and that in this problem the superposition principle works for small perturbations [8], we can omit r_p. The momenta feel only relativistic corrections, so there one can substitute there the classic expressions. Then, substituting the radius vector of the test body

$$\mathbf{r}_3 = r_{\rm kep}\left(\mathbf{i}\cos\omega_3 t + \mathbf{j}\sin\omega_3 t\right), \qquad (15)$$

and for the second body

$$\mathbf{r}_2 = r_2\left(\mathbf{i}\cos\omega_2 t + \mathbf{j}\sin\omega_2 t\right), \qquad (16)$$

and the momenta as derived from them, multiplied by the corresponding masses, and integrating over the period T,

$$\overline{\dot{\mathbf{M}}} = \frac{1}{T}\int_0^T \left(\dot{\mathbf{M}}_{\rm kep} + \dot{\mathbf{M}}_{\rm rel}\right) dt, \qquad (17)$$

we obtain the evolutionary equations of motion

$$\overline{\dot{\mathbf{M}}} = \overline{\dot{\mathbf{M}}}_{\rm kep} + \overline{\dot{\mathbf{M}}}_{\rm rel}. \qquad (18)$$

The average from the Keplerian motion is zero,

$$\overline{\dot{\mathbf{M}}}_{\rm kep} = 0. \qquad (19)$$

Next, integrating the relativistic component, we also get zero,

$$\overline{\dot{\mathbf{M}}}_{\rm rel} = 0. \qquad (20)$$

The orbital stability of the test body motion by definition means vanishing of the mean angular momentum change. As can be seen from this expression, in GR in the first approximation the circular motion of a test body in the orbital plane of the second body in the restricted circular three-body problem is stable.

4. CONCLUSION

We have investigated the relativistic equation of translational motion of a test body in the field of two bodies in GR mechanics, corresponding to the circular restricted three-body problem.

The evolutionary equations of motion have been studied by asymptotic methods of adiabatic theory, through the process of averaging of the corresponding equations using the vector elements $\mathbf{M}$ (the orbital moment) and $\mathbf{A}$ (the Laplace vector), The result shows than in this case the circular orbit of a test body is stable.

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