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To the boundary value problem of ordinary differential equations

Serikbay Aisagaliev and **Zhanat Zhunussova**✉

Institute of Mathematics and Mechanics, Al-Farabi Kazakh National University,
Al-Farabi Avenue 71, Almaty, Kazakhstan

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Abstract. A method for solving of a boundary value problem for ordinary differential equations with boundary conditions at phase and integral constraints is proposed. The base of the method is an immersion principle based on the general solution of the first order Fredholm integral equation which allows to reduce the original boundary value problem to the special problem of the optimal equation.

Keywords: boundary value problem of ordinary differential equations, the first order Fredholm integral equation, the principle of immersion, optimal control problem, optimization problem.

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1 Problem statement

We consider the following boundary value problem

$$\dot{x} = A(t)x + B(t)f(x,t) + \mu(t), \qquad t \in I = [t_0, t_1] \tag{1.1}$$

with boundary conditions

$$(x(t_0) = x_0,\ x(t_1) = x_1) \in S \subset R^{2n}, \tag{1.2}$$

with phase constraints

$$x(t) \in G(t) : G(t) = \{x \in R^n \mid \gamma(t) \le F(x,t) \le \delta(t),\ t \in I\}, \tag{1.3}$$

and integral constraints

$$3g_j(x) \le c_j, \qquad j = \overline{1, m_1}; \tag{1.4}$$

$$g_j(x) = c_j, \qquad j = \overline{m_1 + 1, m_2}; \tag{1.5}$$

$$g_j(x) = \int_{t_0}^{t_1} f_{0j}(x(t), t)\, dt, \qquad j = \overline{1, m_2}; \tag{1.6}$$

✉ Corresponding author. Email: zhzhkh@mail.ru

Here $A(t)$, $B(t)$ are prescribed matrices with piecewise continuous elements of $n \times n$, $n \times m$ order, respectively, $\mu(t)$, $t \in I$ is given n-dimensional vector-function with piecewise continuous elements, m-dimensional vector-function $f(x,t)$ is defined and continuous in the variables $(x,t) = R^n \times I$ and satisfies the following conditions:

$$|f(x,t) - f(y,t)| \le l|x-y|, \qquad \forall(x,t), \qquad (y,t) \in R^n \times I, \qquad l = \text{const} > 0,$$
$$|f(x,t)| \le c_0|x| + c_1(t), \qquad c_0 = \text{const} \ge 0, \qquad c_1(t) \in L_1(I, R^1),$$

S is a convex closed set. Function $F(x,t) = (F_1(x,t), \dots, F_r(x,t))$, $t \in I$ is an r-dimensional vector-function which is continuous in arguments, $\gamma(t) = (\gamma_1(t), \dots, \gamma_r(t))$ and $\delta(t) = (\delta_1(t), \dots, \delta_r(t))$, $t \in I$ are prescribed continuous functions.

The values c_j, $j = \overline{1, m_2}$ are prescribed constants, $f_{0j}(x,t)$, $j = \overline{1, m_2}$ are given continuous functions satisfying to the conditions

$$|f_{0j}(x,t) - f_{0j}(y,t)| \le l_j|x-y|, \qquad \forall(x,t), \qquad (y,t) \in R^n \times I, \qquad j = \overline{1, m_2};$$
$$|f_{0j}(x,t)| \le c_{0j}|x| + c_{1j}(t), \qquad c_{0j} = \text{const}, \qquad c_{1j} \in L_1(I, R^1), \qquad j = \overline{1, m_2}.$$

Note, that: 1) if $A(t) \equiv 0$, $m = n$, $B(t) = I_n$, then the equation (1.1) can be written as

$$\dot{x} = f(x,t) + \mu(t) = \overline{f}(x,t), \qquad t \in I. \tag{1.7}$$

Therefore, the results obtained below remain valid for the equation (1.7) at conditions (1.2)–(1.6);

2) if $f(x,t) = x + \mu_1(t)$ (or $f(x,t) = C(t)x + \mu_1(t)$), then the equation (1.1) can be written in form

$$\dot{x} = A(t)x + B(t)x + B(t)\mu_1(t) + \mu(t) = \overline{A}(t)x + \overline{\mu}(t), \qquad t \in I, \tag{1.8}$$

where $\overline{A}(t) = A(t) + B(t)$, $\overline{\mu}(t) = B(t)\mu_1(t) + \mu(t)$. It follows that the equation (1.8) is a particular case of equation (1.1).

The following problems are stated.

Problem 1. To find necessary and sufficient conditions for the existence of solutions of boundary value problem (1.1)–(1.6).

Problem 2. To construct a solution of boundary value problem (1.1)–(1.6).

As it follows from the problem statement, it is necessary to prove the existence of the pair $(x_0, x_1) \in S$ such that the solution of (1.1) proceeded from the point x_0 at the time t_0 passes through the point x_1 at the time t_1, along with the solution of the system (1.1) for each time the phase constraint is satisfied (1.3), and integrals (1.6) satisfy (1.4), (1.5). In particular, the set S is defined by the relation

$$S = \left\{ (x_0, x_1) \in R^{2n} \mid H_j(x_0, x_1) \le 0,\ j = \overline{1, p};\ \langle a_j, x_0 \rangle + \langle b_j, x_1 \rangle - d_j = 0,\ j = \overline{p+1, s} \right\},$$

where $H_j(x_0, x_1)$, $j = \overline{1, p}$ are convex functions in the variables (x_0, x_1), $x_0 = x(t_0)$, $x_1 = x(t_1)$, $a_j \in R^n$, $b_j \in R^n$, $d_j \in R^1$, $j = \overline{p+1, s}$ are given vectors and numbers, $\langle \cdot, \cdot \rangle$ is the scalar product.

In many cases, in practice the process under study is described by the equation of the form (1.1) in the phase space of the system defined by the phase constraint of the form (1.3). Outside this domain the process is described by completely different equations or the process under investigation does not exist. In particular, such phenomena take place in the research of dynamics of nuclear and chemical reactors (outside the domain (1.3) reactors do not exist.)