IDEALS WITHOUT MINIMAL ELEMENTS IN ROGERS SEMILATTICES

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Аннотация

We prove a criterion for the existence of a minimal numbering, which is reducible to a given numbering of an arbitrary set. The criterion is used to show that, for any infinite A-computable family F of total functions, where empty set' <=(T) the Rogers semilattice R-A(F) of A-computable numberings for F contains an ideal without minimal elements.

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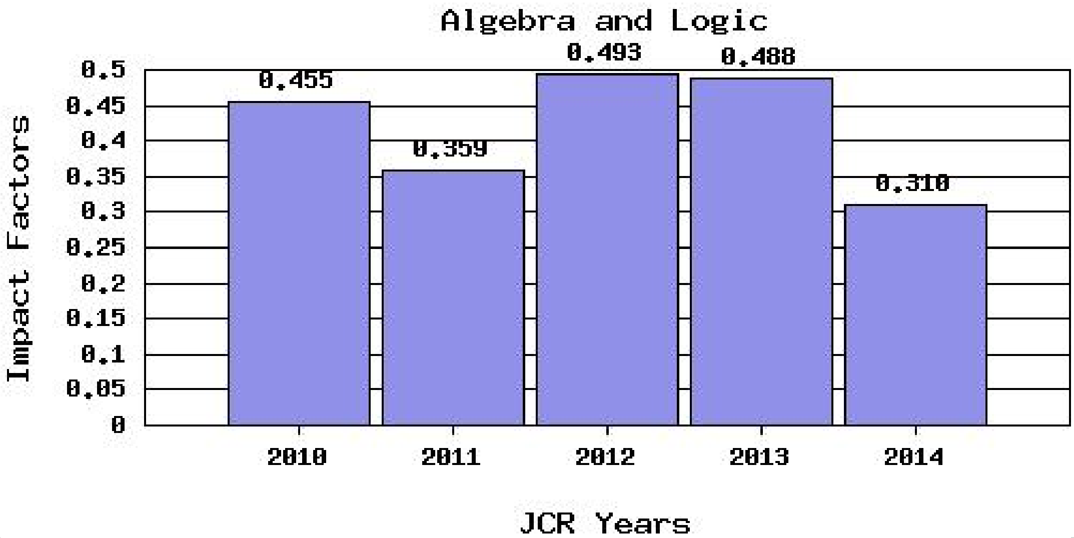

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2014 Impact Factor

Cites in 2014 to articles published in:	2013 = 11	Number of articles published in:	2013 = 43
	2012 = 15		2012 = 41
	Sum: 26		Sum: 84

Calculation: Cites to recent articles / Number of recent articles = 26 / 84 = **0.310**

2013 Impact Factor

Cites in 2013 to articles published in:	2012 = 11	Number of articles published in:	2012 = 41
	2011 = 28		2011 = 39
	Sum: 39		Sum: 80

Calculation: Cites to recent articles / Number of recent articles = 39 / 80 = **0.488**

2012 Impact Factor

Cites in 2012 to articles published in:	2011 = 18	Number of articles published in:	2011 = 39
	2010 = 17		2010 = 32
	Sum: 35		Sum: 71

Calculation:	Cites to recent articles / Number of recent articles	35 / 71 = **0.493**

2011 Impact Factor

Cites in 2011 to articles published in:		Number of articles published in:	
2010	= 10	2010	= 32
2009	= 13	2009	= 32
Sum:	23	Sum:	64

Calculation:	Cites to recent articles / Number of recent articles	23 / 64 = **0.359**

2010 Impact Factor

Cites in 2010 to articles published in:		Number of articles published in:	
2009	= 14	2009	= 32
2008	= 16	2008	= 34
Sum:	30	Sum:	66

Calculation:	Cites to recent articles / Number of recent articles	30 / 66 = **0.455**

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IDEALS WITHOUT MINIMAL ELEMENTS IN ROGERS SEMILATTICES

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Keywords: *minimal numbering, A-computable numbering, Rogers semilattice, ideal.*

We prove a criterion for the existence of a minimal numbering, which is reducible to a given numbering of an arbitrary set. The criterion is used to show that, for any infinite A-computable family F of total functions, where $\varnothing' \leq_T A$, the Rogers semilattice $\mathcal{R}_A(F)$ of A-computable numberings for F contains an ideal without minimal elements.

In this paper, we study generalized computable families of total functions and their generalized computable numberings. Also attempts are made to find elementary properties of corresponding Rogers semilattices, which differ from properties of classical Rogers semilattices for families of computable functions. Families of total A-computable functions, where $\varnothing' \leq_T A$, will be considered.

A surjective mapping of the set ω of all natural numbers onto the class of constructive objects under investigation is called a *numbering* of this class. (By constructive objects are meant those admitting a formal description in some language endowed with a given Gödel numbering of formulas [1].) Let S be a family of computably enumerable (c.e.) sets; then a numbering $\alpha : \omega \to S$ is said to be *computable* if $G_\alpha \rightleftharpoons \{(x, m) \mid x \in \alpha(m)\}$ is a c.e. set. Identifying functions with their graphs, we may assert that for a family F of partial computable functions, a numbering $\alpha : \omega \to F$ is computable if the binary function $g_\alpha \rightleftharpoons \lambda x \lambda y[\alpha(x)](y)$ is partial computable. Let $\alpha : \omega \to S$ and $\beta : \omega \to S$ be two numberings of a same set S. We say that the numbering α is *reducible* to the numbering β if there is a computable function f such that $\alpha = \beta f$, and we write $\alpha \leq \beta$. If $\alpha \leq \beta$ and $\beta \leq \alpha$, then the numberings α and β are said to be *equivalent*, written $\alpha \equiv \beta$. Denote by $\deg(\alpha)$ the degree of α, i.e., the set $\{\beta \mid \beta \equiv \alpha\}$ of numberings. The reducibility of numberings is a preorder relation on the set of all computable numberings of a family S, which we denote by $\mathrm{Com}(S)$. The relation induces a partial order relation on a set of degrees of the numberings in $\mathrm{Com}(S)$, which we also denote by $\leq$. The partially ordered set $\mathcal{R}(\mathrm{S}) = \langle \{\deg(\alpha) \mid \alpha \in \mathrm{Com(S)}\}, \leq \rangle$ is an upper semilattice, which we call the *Rogers semilattice of the family S*.

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A numbering α of a set S is *minimal* if, for any numbering β of S, $\beta \leq \alpha$ implies that $\alpha \leq \beta$. A computable one-to-one numbering is called a *Friedberg* numbering. The *numerical equivalence* θ_α of a numbering α is defined as follows: $\theta_\alpha \rightleftharpoons \{(x, y) \mid \alpha(x) = \alpha(y)\}$. An equivalence relation ε is said to be *positive* if ε is c.e.. By $[W]_\varepsilon$ we denote the set of all numbers which are ε-equivalent to some element of W.

For the notation and notions undefined in the paper, we refer the reader to [1, 2].

Let F be a family of total functions which are computable by an oracle A. A numbering $\alpha : \omega \to F$ is said to be A-*computable* if the binary function $\alpha(n)(x)$ is A-computable [3]. A family F is A-*computable* if it has an A-computable numbering. If A is a recursive set, then we deal with a family of computable functions and its classical computable numberings. A partially ordered set $\mathcal{R}_A(F) = \langle\{\deg(\alpha) \mid \alpha \in \mathcal{C}_A(F)\}, \leq\rangle$, where $\mathcal{C}_A(F)$ is the set of all A-computable numberings of a family F, is called the *Rogers semilattice of the family* F [3, 4].

It is known that in the classical case, the Rogers semilattice of a computable family F either consists of one element or is infinite [5]. In the nontrivial case, the Rogers semilattice of F is not a lattice; moreover, it has no greatest element and contains either one or infinitely many minimal elements [6].

For A-computable numberings of families of total functions, where $\varnothing' \leq_T A$, we have the following results.

THEOREM 1. Let F be an infinite A-computable family of total functions. Then F has infinitely many pairwise nonequivalent A-computable Friedberg numberings.

THEOREM 2. There are an A-computable family F and an A-computable numbering α of the family F such that there is no A-computable Friedberg numbering of F which is reducible to α.

THEOREM 3. If an A-computable family F contains at least two functions, then F has no A-computable principal numbering.

Proofs of the above results are obtained by obvious generalizations of the proofs for corresponding results in [7, 8], which were originally announced in [9] where Theorems 1-3 are proved for specific oracles $A = \varnothing^{n+1}$, $n \in \omega$.

There are well-known examples of infinite families of c.e. sets whose Rogers semilattices contain ideals without minimal elements, for instance, the family of all c.e. sets [10]. Moreover, there is a computable family of c.e. sets whose Rogers semilattice lacks minimal elements altogether [11, 12].

In contrast to the case of families of c.e. sets, for each computable numbering α of an infinite family F of computable functions, there exists a Friedberg numbering of the family F which is reducible to α [5]. This means that the Rogers semilattice of any computable family of total functions does not contain ideals without minimal elements.

Below we show that this result defies generalization if we consider A-computable families and numberings for $\varnothing' \leq_T A$ in place of computable families and numberings. In fact, for any A-computable family F, there is an A-computable numbering of F with no minimal numbering below it. In order to state this, we first extend Badaev's criterion for minimality of a numbering to a criterion for a numbering not bounding any minimal numbering. Note that these criteria hold for numberings of any sets, not only for numberings of families of functions.

THEOREM 4 [12]. Let $\alpha : \omega \to S$ be a numbering of an arbitrary set S. Then the following statements are equivalent:

(a) α is a minimal numbering;

(b) for any c.e. set $R \subseteq \omega$ such that $[R]_{\theta_\alpha} = \omega$, there exists a computable function $G(x)$ such that $(x, G(x)) \in \theta_\alpha$ and $G(x) \in R$ for any $x \in \omega$;

(c) for any c.e. set $R \subseteq \omega$ such that $[R]_{\theta_\alpha} = \omega$, there exists a positive equivalence relation $\varepsilon \subseteq \theta_\alpha$ such that $[R]_\varepsilon = \omega$.

THEOREM 5. Let α be a numbering of an arbitrary set S. Then S has a minimal numbering which is reducible to α if and only if there exists a c.e. set W for which the following conditions hold:

(a) $\alpha(W) = S$;

(b) for any c.e. set $V \subseteq W$, where $\alpha(V) = S$, there is a positive equivalence relation ε such that $\varepsilon \restriction W \subseteq \theta_\alpha$ and $W \subseteq [V]_\varepsilon$.

Here $\varepsilon \restriction W$ is an equivalence relation defined by setting

$$(x, y) \in \varepsilon \restriction W \Leftrightarrow (x \notin W \,\&\, y \notin W \,\&\, x = y) \vee (x \in W \,\&\, y \in W \,\&\, (x, y) \in \varepsilon).$$

Proof. Sufficiency. Assume that W is a c.e. set for which $\alpha(W) = S$ and condition (b) holds. Let f be a computable function enumerating W, i.e., $\mathrm{rng}(f) = W$. It suffices to show that $\beta = \alpha f$ is a numbering of S and that it is minimal. The former claim follows immediately from

$$\beta(\omega) = \alpha f(\omega) = \alpha(W) = S.$$

To prove the latter, we fix a numbering γ of S such that $\gamma \leq \beta$. Let $\gamma = \beta g$ for some computable function g. It suffices to point out a computable function h such that $\beta \leq \gamma$ via h.

Let $V = \mathrm{rng}(f \circ g)$. Then V is c.e. and $V \subseteq \mathrm{rng}(f) = W$. Since γ is a numbering of S and $\gamma = \beta g = \alpha f g$, we have

$$S = \gamma(\omega) = \beta(\mathrm{rng}(g)) = \alpha(\mathrm{rng}(f \circ g)) = \alpha(V).$$

By condition (b), therefore, there is a c.e. equivalence relation ε such that $\varepsilon \restriction W \subseteq \theta_\alpha$ and $W \subseteq [V]_\varepsilon$. This, combined with $W = \mathrm{rng}(f)$ and $V = \mathrm{rng}(f \circ g)$, implies that for any number x there is a number y such that $(f(x), f(g(y))) \in \varepsilon$. Since ε is c.e., while f and g are computable, there is a computable function h for which $(f(x), f(g(h(x)))) \in \varepsilon$. We have $\varepsilon \restriction \mathrm{rng}(f) = \varepsilon \restriction W \subseteq \theta_\alpha$; therefore, $\alpha(f(x)) = \alpha(f(g(h(x))))$. Now $\beta = \alpha f$ and $\gamma = \alpha f g$ imply that $\beta(x) = \gamma(h(x))$ for all x. Hence $\beta \leq \gamma$ via h.

Necessity. Suppose that β is a minimal numbering of the set S that is reducible to the given numbering α. Fix a computable function f such that $\beta = \alpha f$. Let W be the range of f. Then W

has the required properties. Indeed, W is c.e. Since β is a numbering of S, we have

$$S = \beta(\omega) = \alpha f(\omega) = \alpha(W).$$

Hence condition (a) holds. To verify (b), we fix an arbitrary c.e. set V such that $V \subseteq W$ and $\alpha(V) = S$. It suffices to define a c.e. equivalence relation ε for which $\varepsilon \restriction W \subseteq \theta_\alpha$ and $W \subseteq [V]_\varepsilon$.

Let h be a computable function enumerating V and γ a numbering which is reducible to α via h. Since $\alpha(V) = S$, it follows that $\gamma(\omega) = \alpha h(\omega) = \alpha(V) = S$. Hence γ is a numbering of S. Moreover, $V \subseteq W$, while h and f are computable functions. Consequently, the function g defined as $g(x) = \mu_y\,(h(x) = f(y))$ is total and computable. We have $\gamma(x) = \alpha(h(x)) = \alpha(f(g(x))) = \beta(g(x))$, and so $\gamma \leq \beta$ via g. Since β is minimal among the numberings of S, we may assert that $\beta \leq \gamma$. Hence $\beta = \gamma\hat{g}$ for some computable function $\hat{g}$. Then

$$\alpha(f(z)) = \beta(z) = \gamma(\hat{g}(z)) = \alpha(h(\hat{g}(z))) \tag{1}$$

for all z.

Now let ε be the reflexive transitive closure of a relation $\hat{\varepsilon}$ defined by setting

$$x\,\hat{\varepsilon}\,y \Leftrightarrow \exists\, z\,([x = f(z)\,\&\,y = h(\hat{g}(z))] \,\vee\, [x = h(\hat{g}(z))\,\&\,y = f(z)])$$

(i.e., $x\,\varepsilon\,y$ if $x = y$ or there are numbers $n \geq 1$ and $x_0, \dots, x_n$ such that $x = x_0$, $x_i \hat{\varepsilon} x_{i+1}$ for $i < n$ and $x_n = y$).

Obviously, $\hat{\varepsilon}$ is c.e. and symmetric. These properties are preserved under the reflexive transitive closure; therefore, ε is a positive equivalence relation. Moreover, since θ_α is an equivalence relation, we have $\varepsilon \restriction W \subseteq \theta_\alpha$ in view of (1) and $\hat{\varepsilon} \restriction W \subseteq \theta_\alpha$. In order to verify that $W \subseteq [V]_\varepsilon$, we fix an arbitrary $x \in W$. For a unique z such that $f(z) = x$, we then have $x\hat{\varepsilon}y$ with $y = h(\hat{g}(z)) \in V$. □

THEOREM 6. Let F be an infinite A-computable family of total functions, where $\varnothing' \leq_T A$. Then the Rogers semilattice $\mathcal{R}_A(F)$ contains an ideal without minimal elements.

Proof. By Theorem 5, it is sufficient to build an A-computable numbering α of the family F which satisfies the following condition:

$$\forall\, i\,(\alpha(W_i) = \alpha(\omega) \to \exists V_i \subseteq W_i(\alpha(V_i) = \alpha(\omega)\,\&\,\forall \varepsilon_j(\varepsilon_j \restriction W_i \nsubseteq \theta_\alpha \vee W_i \nsubseteq [V_i]_{\varepsilon_j}))), \tag{2}$$

where $W_0, W_1, W_2, \dots$ and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$ are the standard numberings of the families of all c.e. sets and all positive equivalence relations, respectively, and $V_0, V_1, V_2, \dots$ is some sequence of c.e. sets, which will be constructed so that $V_i \subseteq W_i$ for all $i \in \omega$.

First we construct a numbering $\hat{\alpha}$ of a family $\hat{F}$ of constant functions satisfying condition (2).

For brevity, we denote by $a_{i,j}$ and $b_{i,j}$ the values $2\langle i, j\rangle$ and $2\langle i, j\rangle + 1$, respectively, where $\langle i, j\rangle$ is the Cantor number of a pair (i, j). For i fixed and $j = 0, 1, 2, \dots$ arbitrary, we will consistently ask the oracle $\varnothing'$ about whether numbers $a_{i,j}$ and $b_{i,j}$ belong to the set W_i.

(a) If $a_{i,j} \notin W_i$ and $b_{i,j} \notin W_i$ for some j, then we construct

$$\hat{\alpha}(a_{i,j})(x) = 3i,\;\; \hat{\alpha}(b_{i,j})(x) = 3i,\;\; x = 0, 1, 2, \dots.$$

(b) If $a_{i,j} \in W_i$ and $b_{i,j} \notin W_i$ for some j, then we construct

$$\hat{\alpha}(a_{i,j})(x) = 3i + 1,\ \hat{\alpha}(b_{i,j})(x) = 3i,\ x = 0, 1, 2, \ldots.$$

(c) If $a_{i,j} \notin W_i$ and $b_{i,j} \in W_i$ for some j, then we construct

$$\hat{\alpha}(a_{i,j})(x) = 3i,\ \hat{\alpha}(b_{i,j})(x) = 3i + 1,\ x = 0, 1, 2, \ldots.$$

(d) Finally, if $a_{i,j} \in W_i$ and $b_{i,j} \in W_i$ for some j, then we construct $\hat{\alpha}(a_{i,j})$ and $\hat{\alpha}(b_{i,j})$ as follows:

We ask the oracle $\varnothing'$ questions as to whether or not the pair $(a_{i,j}, b_{i,j})$ belongs to ε_j: if $(a_{i,j}, b_{i,j}) \in \varepsilon_j$, then we put

$$\hat{\alpha}(a_{i,j})(x) = 3i + 1,\ \hat{\alpha}(b_{i,j})(x) = 3i + 2,\ x = 0, 1, 2, \ldots;$$

if $(a_{i,j}, b_{i,j}) \notin \varepsilon_j$, then we put

$$\hat{\alpha}(a_{i,j})(x) = 3i + 1,\ \hat{\alpha}(b_{i,j})(x) = 3i + 1,\ x = 0, 1, 2, \ldots.$$

Now let $V_i = (W_i \backslash U_i) \cup \{b_{i,j} \mid (a_{i,j}, b_{i,j}) \in \varepsilon_j\}$, where $U_i = \{x \mid x = a_{i,j},\ j \geq 1 \vee x = b_{i,j},\ j \geq 0\}$ is a computable set.

We verify condition (2) for the numbering $\hat{\alpha}$.

For fixed i in (2), there are two cases to consider:

Case 1. Assume that for some j, one of the numbers $a_{i,j}$ or $b_{i,j}$ does not belong to W_i; i.e., $\hat{\alpha}(a_{i,j}) = 3i$ or $\hat{\alpha}(b_{i,j}) = 3i$. By the construction of $\hat{\alpha}$, only numbers which do not belong to W_i produce a constant function $3i$; therefore, $\hat{\alpha}(W_i) \neq \hat{\alpha}(\omega)$. Consequently, we do not need to check condition (2) for such i (or formally assume that condition (2) holds for i).

Case 2. Suppose that the hypothesis of Case 1 is false for all j (i.e., for all j, $a_{i,j}$ and $b_{i,j}$ belong to W_i). For every such j in (2), we have two options:

Case 2.1. Let $(a_{i,j}, b_{i,j}) \in \varepsilon_j$. Then $\hat{\alpha}(a_{i,j}) = 3i + 1 \neq 3i + 2 = \hat{\alpha}(b_{i,j})$ by construction, and hence $(a_{i,j}, b_{i,j}) \notin \theta_{\hat{\alpha}}$. Therefore, $\varepsilon_j \restriction W_i \not\subseteq \theta_{\hat{\alpha}}$.

Case 2.2. Let $(a_{i,j}, b_{i,j}) \notin \varepsilon_j$. Then $\hat{\alpha}(a_{i,j}) = 3i + 1 = \hat{\alpha}(b_{i,j})$ by construction. The set V_i contains a unique $\hat{\alpha}$-number $a_{i,0}$ of a constant function $3i+1$; i.e., $\hat{\alpha}(z) \neq 3i+1$ for all $z \in V_i \backslash a_{i,0}$. Since ε_j is an equivalence relation and $(a_{i,j}, b_{i,j}) \notin \varepsilon_j$, it follows that $a_{i,j} \notin [a_{i,0}]_{\varepsilon_j}$ or $b_{i,j} \notin [a_{i,0}]_{\varepsilon_j}$. If $a_{i,j} \in [V_i \backslash a_{i,0}]_{\varepsilon_j}$ or $b_{i,j} \in [V_i \backslash a_{i,0}]_{\varepsilon_j}$, then $\varepsilon_j \restriction W_i \not\subseteq \theta_{\hat{\alpha}}$. If $a_{i,j} \notin [V_i \backslash a_{i,0}]_{\varepsilon_j}$ and $b_{i,j} \notin [V_i \backslash a_{i,0}]_{\varepsilon_j}$, then $a_{i,j}$ in W_i or $b_{i,j}$ in W_i does not belong to $[V_i]_{\varepsilon_j}$, which means that $W_i \not\subseteq [V_i]_{\varepsilon_j}$.

Hence condition (2) holds in any case, and it is easy to see that $\hat{\alpha}$ is A-computable by construction, if $\varnothing' \leq_T A$.

Now we replace constant functions in $\hat{F}$ by functions in F and construct a numbering α of F satisfying the following condition:

$$\alpha(x) = \alpha(y) \Leftrightarrow \hat{\alpha}(x) = \hat{\alpha}(y) \tag{3}$$

for all $x, y \in \omega$.

By Theorem 1, an infinite A-computable family F of total functions has an A-computable Friedberg numbering, which we denote by β.

We describe a construction for α.

Step 0. Let $\alpha(0) = \beta(0)$.

Step $s+1$. If $\hat{\alpha}(s+1) = \hat{\alpha}(t)$ for some $t \leq s$, then $\alpha(s+1) = \alpha(t)$ for the least such t. Otherwise, $\alpha(s+1) = \beta(k)$ for the least k for which $\alpha(t) \neq \beta(k)$ with all $t \leq s$.

An equality of A-computable constant functions is an A-computable relation. Therefore, the numbering α is A-computable. It is not hard to see that α satisfies condition (3), which guarantees that α, as well as $\hat{\alpha}$, will satisfy condition (2). □

In [13], we announced the following: If A is an arbitrary set and F is an infinite A-computable family of total functions, which has (up to equivalence) two A-computable Friedberg numberings, then F has infinitely many such Friedberg numberings. Also if $\varnothing <_T A <_T \varnothing'$ and an A-computable family F of total functions contains at least two functions, then F has no A-computable principal numbering. For the case where A is Turing incomparable with $\varnothing'$, the question about the existence of a principal numbering is still open.

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