PHYSICAL REVIEW E **85**, 015204(R) (2012)

Experiments on oscillator ensembles with global nonlinear coupling

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(Received 16 November 2011; published 31 January 2012)

We experimentally analyze collective dynamics of a population of 20 electronic Wien-bridge limit-cycle oscillators with a nonlinear phase-shifting unit in the global feedback loop. With an increase in the coupling strength we first observe formation and then destruction of a synchronous cluster, so that the dependence of the order parameter on the coupling strength is not monotonic. After destruction of the cluster the ensemble remains nevertheless coherent, i.e., it exhibits an oscillatory collective mode (mean field). We show that the system is now in a self-organized quasiperiodic state, predicted in Rosenblum and Pikovsky [Phys. Rev. Lett. **98**, 064101 (2007)]. In this state, frequencies of all oscillators are smaller than the frequency of the mean field, so that the oscillators are not locked to the mean field they create and their dynamics is quasiperiodic. Without a nonlinear phase-shifting unit, the system exhibits a standard Kuramoto-like transition to a fully synchronous state. We demonstrate a good correspondence between the experiment and previously developed theory. We also propose a simple measure which characterizes the macroscopic incoherence-coherence transition in a finite-size ensemble.

DOI: 10.1103/PhysRevE.85.015204 PACS number(s): 05.45.Xt, 05.65.+b

Mean-field approximation is widely used in the description of oscillator networks with high degrees of connectivity. Models of oscillator ensembles with mean-field, or global, coupling describe collective dynamics of oscillating objects of various nature, including fireflies, pedestrians on the footbridges, hand-clapping individuals in a large audience, Josephson junctions, lasers, electrochemical oscillators, and neurons, to name just a few [1]. The main effect of the global coupling is the well-understood emergence of collective synchrony, reflected in the increase in the mean-field amplitude with the interaction strength and often referred to as the Kuramoto transition. Further well-known effects are clustering [2] and chaotization of the mean field [3,4]. Experiments on globally coupled oscillators have been performed by Hudson, Kiss, and collaborators [5]. Using an ensemble of 64 electrochemical oscillators, they have confirmed most theoretical predictions. In particular, they have demonstrated the Kuramoto transition in ensembles of periodic and chaotic oscillators. Other laboratory experiments have been conducted with Josephson junctions, photochemical oscillators, and vibrating motors on a common support [6].

A subject of recent interest is coherent though not synchronous states, also denoted as partial synchrony. Such regimes have been observed in networks of pulse coupled integrate-and-fire units [4,7] and in ensembles of Stuart-Landau and phase oscillators with global nonlinear coupling [8]. The latter systems exhibit an interesting transition from synchrony to self-organized quasiperiodicity (SOQ). In the SOQ state the frequency of the mean field differs from the frequency of oscillators, i.e., the emergent collective mode and individual units are not locked. The primary goal of this Rapid Communication is experimental verification of these results. For this purpose, we performed experiments with electronic oscillators, globally coupled via a common feedback loop with a phase-shifting unit. The coupling is nonlinear in the sense that the phase shift depends on the amplitude of the collective oscillation. We demonstrate, with an increase in the strength of the global coupling, a transition from an asynchronous state to collective synchrony and then to SOQ.

Typically the tendency to synchrony increases with the coupling strength. However, in some setups the increase of the coupling parameter makes the initially attractive interaction repulsive, leading to the breakup of synchrony. As a result, the system undergoes a transition either to an asynchronous state or to a state of partial synchrony. In the latter case, the system stays at the border between synchrony and asynchrony and exhibits interesting dynamics, in particular, SOQ states. We discuss the SOQ dynamics using as an illustration a solvable model [8] of $N \gg 1$ identical globally coupled Stuart-Landau oscillators:

$$\dot{a}_k = (1 + i\omega)a_k - |a_k|^2 a_k + e^{i\alpha} A, \quad k = 1, \dots, N, \qquad (1)$$

where α is the phase shift of the coupling. Suppose first a simple mean-field coupling of strength $\varepsilon > 0$, i.e., $A = \varepsilon N^{-1} \sum_{k=1}^{N} a_k$. For $\varepsilon \ll 1$, this system reduces to the well-studied Kuramoto-Sakaguchi model [9]

$$\dot{\varphi}_k = \omega + \varepsilon R \sin(\Theta - \varphi_k + \alpha), \qquad (2)$$

where $\varphi_k = \arg a_k$ and the mean-field phase Θ and amplitude (order parameter) R are determined by $Re^{i\Theta} = N^{-1} \sum_{k=1}^{N} e^{i\varphi_k}$. The dynamics of Eq. (2) is determined by the phase shift α: For $|\alpha| < \pi/2$ the coupling is attractive and the system synchronizes ($R = 1$), otherwise the coupling is repulsive and the system remains asynchronous ($R = 0$). Suppose now that the mean field has its own dynamics, described by

$$\dot{A} = -\gamma A + i\bar{\omega} + i\eta|A|^2 A + \varepsilon N^{-1} \sum_{k=1}^{N} a_k. \qquad (3)$$

For weak coupling and large γ, Eqs. (1) and (3) reduce to

$$\dot{\varphi}_k = \omega + \bar{\varepsilon} R \sin[\Theta - \varphi_k + \beta(R,\bar{\varepsilon})], \qquad (4)$$

where $\bar{\varepsilon} = \varepsilon/\gamma$ and $\beta(R,\bar{\varepsilon}) = \alpha + \eta\gamma^{-1}\bar{\varepsilon}^2 R^2$ [8]. This phase equation differs from Eq. (2) by the dependence of the phase shift on R, ε, which is crucial for the dynamics. Indeed, let $|\alpha| < \pi/2$, then for small $\bar{\varepsilon}$ the phase shift β is also smaller than $\pi/2$, the system synchronizes, and $R = 1$. However, if the coupling increases beyond the critical value $\bar{\varepsilon} > \varepsilon_{\mathrm{cr}} = \gamma(\pi/2 - \alpha)/\eta$, then β becomes larger than $\pi/2$ and, hence, the interaction becomes repulsive. As a result, the system tends to desynchronize and to decrease the order parameter, which would make $\beta < \pi/2$, i.e., the interaction again would become attractive. Finally, the system settles exactly at the border between synchrony and asynchrony, with the order parameter $R < 1$ determined from the condition $\beta(R,\bar{\varepsilon}) = \pi/2$. This desynchronization transition results in the divergence of frequencies of the mean field and of individual oscillators; generally these frequencies are incommensurate, and, hence, the dynamics of oscillators is quasiperiodic [8]. An analytical treatment of the model (4) was extended to the cases of Lorentzian [10] and uniform [11] distribution of frequencies. We briefly discuss the latter case, since it is closer to experimental implementation. With the increase of $\bar{\varepsilon}$ first the transition to synchrony is observed. If the frequency distribution is sufficiently narrow then all oscillators form a synchronous cluster, otherwise part of them remains asynchronous. Next, oscillators, one by one, leave the synchronous cluster, and finally the SOQ state is formed, where all oscillators differ in frequency from the mean field they create. The transition from synchrony to SOQ is accompanied by a decrease in the order parameter. Although the theoretical treatment has been performed for phase oscillators, we expect that this effect can be observed for general limit-cycle oscillators, provided the phase shift in the global feedback monotonically depends on the mean-field amplitude.

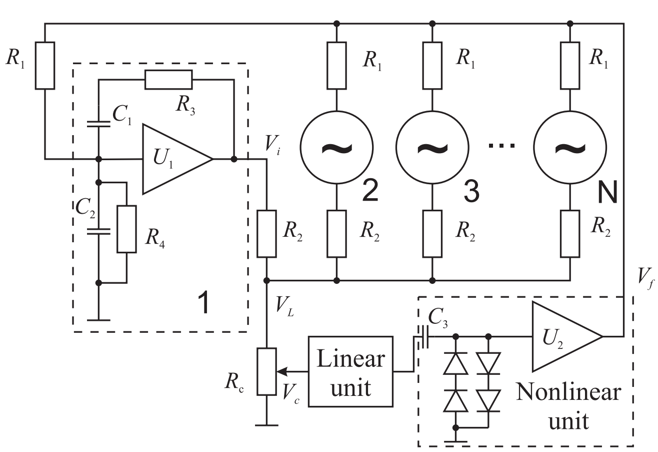

FIG. 1. Scheme of the experimental setup of $N = 20$ globally coupled oscillators. All oscillators have an identical structure and therefore only the first one is shown in detail. The global coupling is organized via the common load R_c. A fraction of the voltage across R_c is fed back to each oscillator via the feedback loop, consisting of a linear (standard RC circuit) and nonlinear phase-shifting units.

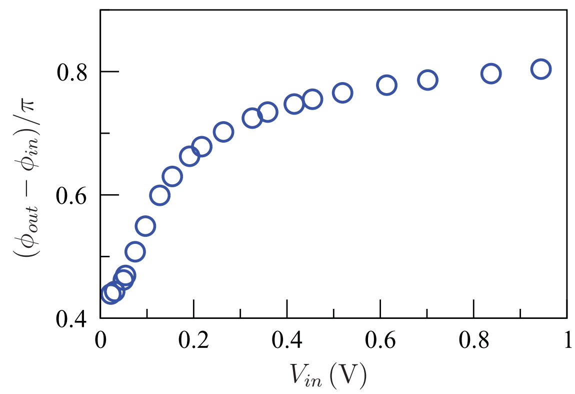

FIG. 2. (Color online) Characteristic of the nonlinear phase-shifting unit: phase shift between the output and input vs the amplitude of the input.

We performed experiments with $N = 20$ electronic generators, coupled via a global feedback loop—see Fig. 1. Coupling is organized via a common load R_c; a fraction of the voltage across this potentiometer is fed to the input of the phase-shifting unit, and the output of the latter, V_f, is fed back to all oscillators via resistors R_1. With the voltage across R_c denoted as V_L, the input voltage to the feedback loop can be

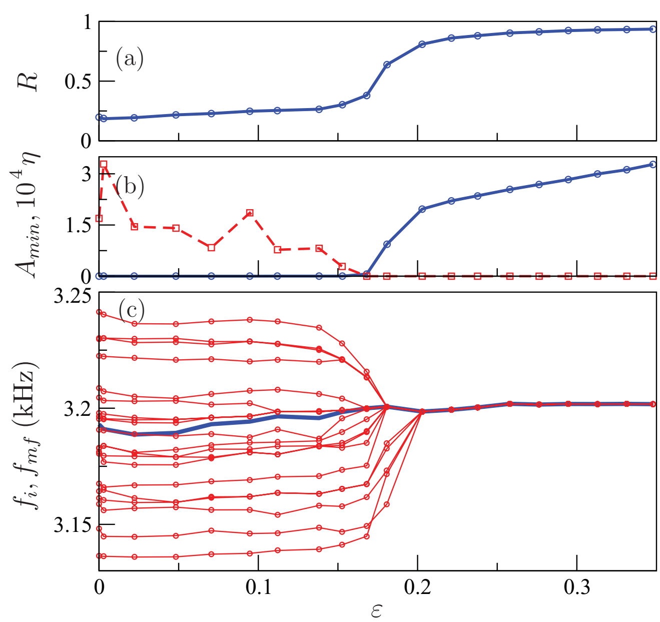

FIG. 3. (Color online) Results of the experiment with the linear phase-shifting unit. Order parameter R (a) and minimal mean-field amplitude A_{min} [(b), blue circles] reveal the synchronization transition at the coupling strength $\varepsilon \approx 0.17$. This is also confirmed by the plot of η [(b), red squares]: This quantity shows that for $\varepsilon \gtrsim 0.17$, the instantaneous frequency of the mean field is always positive, as expected for a coherent, oscillatory mean field. The transition can be also very good seen from the frequency plot in (c): At $\varepsilon \approx 0.17$ several oscillators form a synchronous cluster and for $\varepsilon \gtrsim 0.2$ full frequency locking is observed, with R close to 1. Here the circles show the frequencies of oscillators f_i and the bold blue line shows the mean-field frequency f_{mf}. Notice that for subthreshold coupling R is not small due to the finite-size effect; here A_{mf} is more efficient for the determination of the threshold.

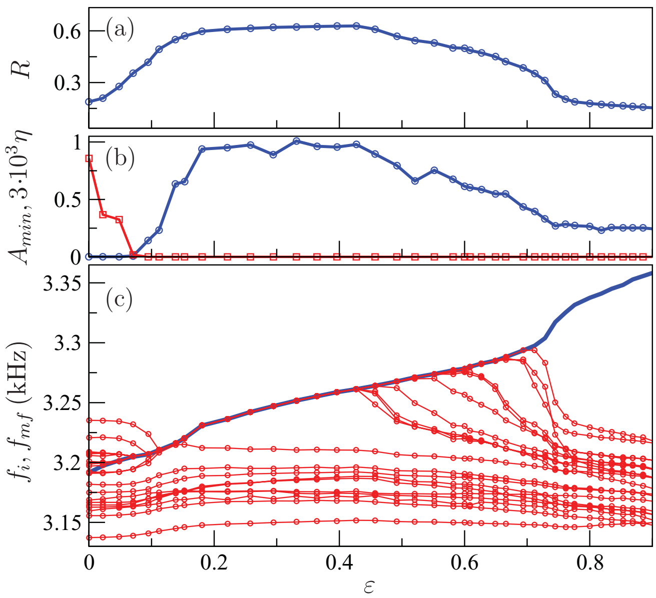

FIG. 4. (Color online) The same as in Fig. 3 but for the experiment with the linear and nonlinear phase-shifting units. At $\varepsilon \approx 0.12$ we observe the transition to a partially synchronous state, where the fastest oscillators lock onto each other and to the mean field. Between $\varepsilon \approx 0.43$ and $\varepsilon \approx 0.72$ the oscillators leave the cluster and for $\varepsilon \gtrsim 0.72$ the SOQ state is observed: The mean field is faster than all oscillators. Although the values of the order parameter in the asynchronous ($\varepsilon \lesssim 0.12$) and SOQ states are almost the same, these states are qualitatively different (see Fig. 5), and can be easily distinguished by the quantities $A_{\min}$ and η (see text).

written as $V_c = \varepsilon V_L$. The parameter ε, $0 \leqslant \varepsilon \leqslant 1$, quantifies the strength of the global coupling. Individual units are Wien-bridge oscillators with saturation of the amplitude due to the negative feedback loop of the operational amplifier (not shown) [12]. The input-output characteristic of the operational amplifier is a sigmoid curve which can be approximated as $V = ku_s \tanh(u/u_s)$, where u and V are the input and output voltages, $u_s = 3.2$ V determines the range of the input voltages where the amplifier works without saturation, and $k = 10$ is the slope of the characteristics in the linear regime. All oscillators were tuned to have approximately the same output voltage $\approx$1 V and close frequencies $\approx$3.1 kHz. The phase-shifting unit has a linear and nonlinear parts. The former is implemented via two standard RC circuits plus amplifiers, and the details of the latter are shown in Fig. 1; the characteristic of the nonlinear part is shown in Fig. 2.

We performed three experiments. In the first one the phase-shifting unit was excluded so that the signal from the common load was directly applied to the inputs of oscillators, i.e., $V_f = V_c$. In the second experiment only the linear phase-shifting unit was included, and in the third run we had both linear and nonlinear units, as shown in Fig. 1. In each experiment we gradually changed the input to the feedback loop V_c from zero to its maximal value V_L and recorded the outputs of all oscillators, V_i, $i = 1,\dots,N$, and the mean-field voltage V_L [13]. In each recording we obtained 10^5 points per channel, with the sampling rate 65 kHz. For each value of the coupling strength $\varepsilon = V_c/V_L$ we performed ten recordings.

For the presentation of results we have computed, for each ε, the following quantities: (i) Instantaneous phases φ_i of all oscillators and the instantaneous phase and amplitude A_{mf} of the mean field V_L were obtained with the help of the Hilbert transform; (ii) frequencies f_i of all oscillators and frequency f_{mf} of the mean field were computed from the unwrapped phases for each recording and then averaged over ten recordings; (iii) the order parameter R was obtained by averaging the quantity $N^{-1}\sum_{j=1}^{N} e^{i\phi_j}$ over time and over ten measurements; (iv) the minimal (over all ten measurements) value $A_{\min}$ of the instantaneous mean-field amplitude A_{mf}; and (v) the fraction η of the data points where the instantaneous frequency of the mean field is negative. Typically, synchronization transition in a globally coupled system is traced by plotting R vs ε. In the limit $N \to \infty$, $R = 0$ in the incoherent state. However, since in our case $N = 20$, the finite-size fluctuations of the mean field in this state are quite large (they are known to scale as $1/\sqrt{N}$) and therefore R is not small either. We find that the distinction between incoherent (fluctuating mean field) and coherent (oscillatory mean field) states can be better revealed by $A_{\min}$ and η (see also the discussion of Fig. 5 below).

In the first and second experiments (no phase-shifting unit and linear unit, respectively), we observed standard Kuramoto transitions to collective synchrony. These transitions occurred at $\varepsilon \approx 0.85$ and $\varepsilon \approx 0.17$, respectively [14] (see Fig. 3, where the second experiment is illustrated), and were characterized by a monotonic dependence of R and $A_{\min}$ on ε. In the third, main, experiment, we observed a nonmonotonic dependence of R and $A_{\min}$ on ε (Fig. 4). We have found that with an increase of ε, ten oscillators formed a cluster at $\varepsilon \approx 0.12$, while the other ten remained asynchronous. Next, the frequency-locked

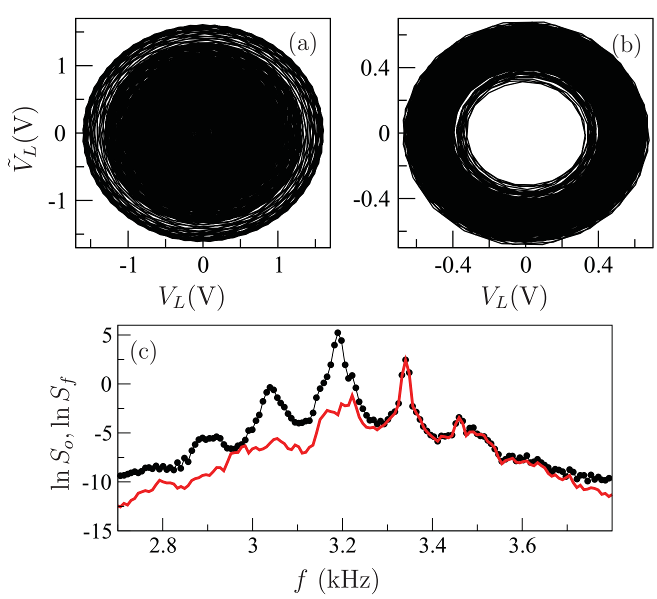

FIG. 5. (Color online) Phase portraits for the mean field. (a) Asynchronous state, $\varepsilon = 0.05$. (b) SOQ state, $\varepsilon = 0.85$ (cf. Fig. 4); $\tilde{V}_L$ is the Hilbert transform of the mean field V_L. The pattern in (a) is typical for a fluctuation process with the amplitude dropping to zero, whereas the portrait in (b) clearly demonstrates coherent, oscillatory mean-field dynamics with the well-defined phase and frequency. (c) Power spectra of one oscillator S_o (circles) and of the mean field S_f (solid line), for $\varepsilon = 0.85$.

oscillators leaved the cluster one by one. Finally, the SOQ state appeared at $\varepsilon \approx 0.72$. In order to show that this is indeed a transition to SOQ but not simply a breakup of synchrony, we plot in Figs. 5(a) and 5(b) the Hilbert transform of the mean field versus the mean field itself. We see that in the asynchronous state the pattern is typical for a narrowband random process, with the amplitude dropping practically to zero, whereas in the SOQ state the mean field is clearly oscillatory and its phase and frequency are well defined. The SOQ dynamics is illustrated by power spectra in Fig. 5(c).

Thus, we have experimentally demonstrated a state where oscillators are synchronized neither with each other nor with the mean field, but the amplitude of the latter is, nevertheless, nonzero. This peculiar coherent state is possible because phases of oscillators, though not locked, are coordinated in a way that their distribution is nonuniform. Our results correspond well to analytical results for phase oscillators [8,11]. The SOQ regime we observe emerges when the system is brought, due to the phase shift, close to the point where attractive interaction becomes repulsive. Thus, we expect SOQ to be observed in other physical systems where the global coupling is characterized by an amplitude-dependent phase shift or time delay. Next, the same dynamics appears in systems with nonlinear coupling, e.g., described by Eq. (1) with $A = c_1 B + c_2 |B|^2 B$, where $B = N^{-1}\sum_{k=1}^{N} a_k$ and $c_{1,2}$ are complex coupling coefficients [8]—cf. Ref. [15]. Moreover, numerical observations, e.g., reported in Ref. [16], indicate that SOQ states can appear in linearly coupled ensembles of strongly nonlinear oscillators. An analysis of such systems is a topic for future theoretical and experimental studies. Finally, we have proposed a simple measure $A_{\min}$ which reliably reveals a macroscopic incoherence-coherence transition; we believe that it can be useful for other studies of the finite-size ensemble, both experimental and numerical studies.

We gratefully acknowledge helpful discussions with Y. Baibolatov, A. Pikovsky, and M. Zaks. V.I.P. was supported by RFFI Grant No. 12-02-00377.

[1] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer, Berlin, 1984); A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization. A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, UK, 2001); D. Golomb, D. Hansel, and G. Mato, in *Neuro-informatics and Neural Modeling*, Handbook of Biological Physics, edited by F. Moss and S. Gielen (Elsevier, Amsterdam, 2001), Vol. 4, pp. 887–968; S. H. Strogatz, Physica D **143**, 1 (2000); *Sync: The Emerging Science of Spontaneous Order* (Hyperion, New York, 2003); E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, UK, 2002), 2nd ed.; S. C. Manrubia, A. S. Mikhailov, and D. H. Zanette, *Emergence of Dynamical Order: Synchronization Phenomena in Complex Systems*, Lecture Notes in Complex Systems, Vol. 2 (World Scientific, Singapore, 2004); J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort, and R. Spigler, Rev. Mod. Phys. **77**, 137 (2005); M. Breakspear, S. Heitmann, and A. Daffertshofer, Front. Human Neurosci. **4**, 190 (2010).

[2] K. Okuda, Physica D **63**, 424 (1993); H. Kori and Y. Kuramoto, Phys. Rev. E **63**, 046214 (2001); Z. Liu, Y. C. Lai, and F. C. Hoppensteadt, *ibid.* **63**, 055201 (2001); Y. Maistrenko, O. Popovych, O. Burylko, and P. A. Tass, Phys. Rev. Lett. **93**, 084102 (2004).

[3] P. C. Matthews and S. H. Strogatz, Phys. Rev. Lett. **65**, 1701 (1990); V. Hakim and W. J. Rappel, Phys. Rev. A **46**, R7347 (1992); N. Nakagawa and Y. Kuramoto, Prog. Theor. Phys. **89**, 313 (1993); Physica D **75**, 74 (1994).

[4] S. Olmi, A. Politi, and A. Torcini, Europhys. Lett. **92**, 60007 (2010).

[5] W. Wang, I. Z. Kiss, and J. L. Hudson, Chaos **10**, 248 (2000); Phys. Rev. Lett. **86**, 4954 (2001); I. Z. Kiss, W. Wang, and J. L. Hudson, Chaos **12**, 252 (2002); I. Z. Kiss, Y. Zhai, and J. L. Hudson, Science **296**, 1676 (2002).

[6] P. Barbara, A. B. Cawthorne, S. V. Shitov, and C. J. Lobb, Phys. Rev. Lett. **82**, 1963 (1999); G. Filatrella, N. F. Pedersen, and K. Wiesenfeld, Phys. Rev. E **61**, 2513 (2000); A. F. Taylor, P. Kapetanopoulos, B. J. Whitaker, R. Toth, L. Bull, and M. R. Tinsley, Phys. Rev. Lett. **100**, 214101 (2008); D. Mertens and R. Weaver, Complexity **16**, 45 (2011).

[7] C. van Vreeswijk, Phys. Rev. E **54**, 5522 (1996); P. Mohanty and A. Politi, J. Phys. A: Math. Gen. **39**, L415 (2006).

[8] M. Rosenblum and A. Pikovsky, Phys. Rev. Lett. **98**, 064101 (2007); A. Pikovsky and M. Rosenblum, Physica D **238**, 27 (2009).

[9] Y. Kuramoto, in *International Symposium on Mathematical Problems in Theoretical Physics*, edited by H. Araki, Lecture Notes in Physics, No. 39 (Springer, New York, 1975), p. 420; H. Sakaguchi and Y. Kuramoto, Prog. Theor. Phys. **76**, 576 (1986).

[10] A. Pikovsky and M. Rosenblum, Physica D **240**, 872 (2011).

[11] Y. Baibolatov, M. Rosenblum, Z. Z. Zhanabaev, and A. Pikovsky, Phys. Rev. E **82**, 016212 (2010).

[12] Components used are as follows: $R_c = 10$ kΩ, $R_1 = 62$ kΩ, $R_2 = 1$kΩ, $R_3 = 1.1$ kΩ, $R_4 = 1.1$ kΩ (trimmer potentiometer was used to tune the oscillator frequency), $C_{1,2} = 51$ nF, $C_3 = 100$ nF, operational amplifiers LM741, diodes 1N2283.

[13] It can be easily shown that $V_L = N^{-1}\sum_{j=1}^{N} V_j$, provided $R_2 \ll N R_c$.

[14] This agrees with the theoretically known effect of phase shift (or time delay) on the synchronization threshold. In order to get insight as to why the phase shift in our setup enhances synchrony, we have derived the equations of the system. We obtained the van der Pol-type equations, driven by the common force $\sim(\frac{V_f}{R_3 C_1} + \dot{V}_f)$. Because of the combination of the mean field and its derivative, the zero phase shift is not optimal.

[15] V. García-Morales, A. Orlov, and K. Krischer, Phys. Rev. E **82**, 065202 (2010).

[16] A. Vilfan and T. Duke, Phys. Rev. Lett. **91**, 114101 (2003).