Geometrical Features of the Soliton Solution

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Abstract It is well known, that integrable equations are solvable by the inverse scattering method (Ablowitz and Clarkson in Solitons, Non-linear Evolution Equations and Inverse Scattering, 1992). Investigating of the integrable spin equations in $(1 + 1)$, $(2 + 1)$ dimensions are topical both from the mathematical and physical points of view (Lakshmanan and Myrzakulov in J. Math. Phys. 39:3765–3771, 1998; Gardner et al. in Phys. Rev. Lett. 19(19):1095–1097, 1967). Integrable equations admit different kinds of physically interesting solutions as solitons, vortices, dromions etc. We consider an integrable spin M-I equation (Myrzakulov and Vijayalakshmi in Phys. Lett. A 233:391–396, 1997). There is a corresponding Lax representation. And the equation allows an infinite number of integrals of motion. We construct a surface corresponding to soliton solution of the equation. Further, we investigate some geometrical features of the surface.

Keywords Surface · Soliton · Nonlinear equation

1 Introduction

We consider the connection between the surface and the soliton equation M-I which has the form [2],

$$\mathbf{S}_t = (\mathbf{S} \times \mathbf{S}_y + u\mathbf{S})_x, \tag{1.1}$$

$$u_x = -\big(\mathbf{S}, (\mathbf{S}_x \times \mathbf{S}_y)\big), \tag{1.2}$$

where $\mathbf{S}$ is spin vector, $S_1^2 + S_2^2 + S_3^2 = 1$, $\times$ is vector product, u is a scalar function. We identify the spin vector $\mathbf{S}$ and vector $\mathbf{r}_x$ according to [2]

$$\mathbf{S} \equiv \mathbf{r}_x. \tag{1.3}$$

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Then (1.1), (1.2) take the form

$$\mathbf{r}_{xt} = (\mathbf{r}_x \times \mathbf{r}_{xy} + u\mathbf{r}_x)_x, \tag{1.4}$$

$$u_x = -\big(\mathbf{r}_x, (\mathbf{r}_{xx} \times \mathbf{r}_{xy})\big). \tag{1.5}$$

If we integrate (1.4) by x, then it takes the form

$$\mathbf{r}_t = \mathbf{r}_x \times \mathbf{r}_{xy} + u\mathbf{r}_x. \tag{1.6}$$

Taking into account Gauss–Weingarten equation and $E = \mathbf{r}_x^2 = 1$ the system is defined as

$$\mathbf{r}_t = \left(u + \frac{MF}{\sqrt{\Lambda}}\right)\mathbf{r}_x - \frac{M}{\sqrt{\Lambda}}\mathbf{r}_y + \Gamma_{12}^2\sqrt{\Lambda}\mathbf{n}, \tag{1.7}$$

$$u_x = \sqrt{\Lambda}\big(L\Gamma_{12}^2 - M\Gamma_{11}^2\big), \tag{1.8}$$

where

$$\Gamma_{11}^2 = \frac{2EF_x - EE_t - FE_x}{2\Lambda}, \tag{1.9}$$

$$\Gamma_{12}^2 = \frac{EG_x - FE_t}{2\Lambda}, \tag{1.10}$$

$\Lambda = EG - F^2$. M-I equation is integrable equation and has soliton solutions.

2 Construction of Surface Corresponding to Soliton Solution

Here we present the one-soliton solution of (1.1), (1.2) [2],

$$S_3(x, y, t) = 1 - \frac{2\eta^2}{\eta^2 + \xi^2}\operatorname{sech}^2(\chi_{1R}), \tag{2.1}$$

$$S^+(x, y, t) = \frac{2\eta}{\eta^2 + \xi^2}\big[i\xi - \eta th(\chi_{1R})\big]\operatorname{sech}(\chi_{1R}), \tag{2.2}$$

$$\chi_1 = \chi_{1R} + i\chi_{1I}, \qquad \lambda_1 = \eta + i\xi, \tag{2.3}$$

$$m_1 = m_{1R}(\rho) + im_{1I}(\rho), \quad m_j(y, t) = m_j(\rho), \tag{2.4}$$

$$\chi_{1R} = \eta x + m_{1R}(\rho) + c_{1R}, \quad \rho = y + i\lambda_j t, \tag{2.5}$$

$$\chi_{1I} = \xi x + m_{1I}(\rho) + c_{1I}, \quad c = \ln\big(2\eta/\lambda_1^*\big), \tag{2.6}$$

$$m_{1R}(\rho) = \operatorname{Re}\big[m_1(\rho)\big], \qquad m_{1I}(\rho) = \operatorname{Im}\big[m_1(\rho)\big], \tag{2.7}$$

which we use in the following theorem.

Theorem 2.1 (Main Theorem) *One-soliton solution* (2.1)–(2.7) *of the spin system M-I can be represented as components of the vector* $\mathbf{r}_x$, *where*

$$r_1 = \frac{2\eta}{(\eta^2+\xi^2) ch\chi_{1R}} + c_1, \tag{2.8}$$

$$r_2 = \frac{2\xi}{\eta^2+\xi^2} \operatorname{arctg}(sh\chi_{1R}) + c_2, \tag{2.9}$$

$$r_3 = x - \frac{2\eta}{\eta^2+\xi^2} th\chi_{1R} + c_3, \tag{2.10}$$

c_1, c_2, c_3 *are constants. Solution of the form* (2.8)–(2.10) *corresponds to the surface with the following coefficients of the first and second fundamental forms*

$$E = 1, \qquad G = \frac{4m_{1Ry}^2}{(\eta^2+\xi^2) ch^2\chi_{1R}}, \tag{2.11}$$

$$F = \frac{2\eta m_{1Ry}}{(\eta^2+\xi^2) ch^2\chi_{1R}}, \qquad L = \frac{4\eta^3\xi m_{1Ry}}{\sqrt{g}(\eta^2+\xi^2)^2 ch^4\chi_{1R}}, \tag{2.12}$$

$$M = \frac{4\eta^2\xi m_{1Ry}^2}{\sqrt{g}(\eta^2+\xi^2)^2 ch^4\chi_{1R}}, \qquad N = \frac{4\eta\xi m_{1Ry}^3}{\sqrt{g}(\eta^2+\xi^2)^2 ch^4\chi_{1R}}. \tag{2.13}$$

Proof From (1.3) we have

$$(S_1, S_2, S_3) = (r_{1x}, r_{2x}, r_{3x}), \tag{2.14}$$

i.e.

$$r_{1x} = S_1, \qquad r_{2x} = S_2, \qquad r_{3x} = S_3. \tag{2.15}$$

Hence

$$r_1 = \int S_1 dx + c_1, \tag{2.16}$$

$$r_2 = \int S_2 dx + c_2, \tag{2.17}$$

$$r_3 = \int S_3 dx + c_3, \tag{2.18}$$

where c_1, c_2, c_3 are constants of integration. Note

$$S^+ = S_1 + iS_2 = r_x^+, \tag{2.19}$$

then

$$r^+ = r_1 + ir_2 = \int S^+ dx + c^+, \tag{2.20}$$

where c^+ is constant of integration. Substituting (2.1) in (2.18), we have

$$r_3 = \int S_3 dx + c_3 = \int \left[1 - \frac{2\eta^2}{\eta^2+\xi^2}\operatorname{sech}^2(\chi_{1R})\right]dx + c_3$$
$$= x - \frac{2\eta}{(\eta^2+\xi^2)} th(\chi_{1R}) + c_3^*, \tag{2.21}$$

where $c_3^* = c_3 + c_3'$. $c_3 \equiv c_3^*$, then

$$r_3 = x - \frac{2\eta}{(\eta^2+xi^2)} th(\chi_{1R}) + c_3. \tag{2.22}$$

Substituting (2.2) into (2.20) we have

$$r^+ = r_1 + ir_2 = \int S^+ dx + c^+$$
$$= \int \frac{2\eta}{\eta^2+\xi^2}\left[i\xi - \eta th(\chi_{1R})\right]\operatorname{sech}(\chi_{1R})dx + c^+$$
$$= \frac{2i\xi}{\eta^2+\xi^2}\operatorname{arctg}(sh\chi_{1R}) + \frac{2\eta}{\eta^2+\xi^2}\frac{1}{ch\chi_{1R}} + c'' + c^+ + c'''. \tag{2.23}$$

We denote $c_1 = c''$, $c_2 = c^+ + c'''$, then

$$r^+ = \frac{2\eta}{(\eta^2+\xi^2)ch\chi_{1R}} + c_1 + i\left(\frac{2\xi}{\eta^2+\xi^2}\operatorname{arctg}(sh\chi_{1R}) + c_2\right), \tag{2.24}$$

i.e. we have obtained

$$r_1 = \frac{2\eta}{(\eta^2+\xi^2)ch\chi_{1R}} + c_1, \qquad r_2 = \frac{2\xi}{\eta^2+\xi^2}\operatorname{arctg}(sh\chi_{1R}) + c_2. \tag{2.25}$$

Thus, (2.22), (2.25) give us (2.8)–(2.10).

We proceed to prove the second part of the theorem. From (2.22) and (2.25) we have

$$r_{1x} = -\frac{2\eta^2 sh\chi_{1R}}{(\eta^2+\xi^2)ch^2\chi_{1R}}, \qquad r_{2x} = \frac{2\eta\xi}{(\eta^2+\xi^2)ch\chi_{1R}}, \tag{2.26}$$

$$r_{3x} = 1 - \frac{2\eta^2}{(\eta^2+\xi^2)ch^2\chi_{1R}}, \qquad r_{1y} = -\frac{2\eta sh\chi_{1R} m_{1Ry}}{(\eta^2+\xi^2)ch^2\chi_{1R}}, \tag{2.27}$$

$$r_{2y} = \frac{2\xi m_{1Ry}}{(\eta^2+\xi^2)ch\chi_{1Ry}}, \qquad r_{3y} = -\frac{2\eta m_{1Ry}}{(\eta^2+\xi^2)ch^2\chi_{1Ry}}. \tag{2.28}$$

Then we can calculate

$$\begin{aligned} E = \mathbf{r}_x^2 = r_{1x}^2 + r_{2x}^2 + r_{3x}^2 \\ = \frac{4\eta^4 sh^2\chi_{1R}}{(\eta^2+\xi^2)^2 ch^4\chi_{1R}} + \frac{4\eta^2\xi^2}{(\eta^2+\xi^2)^2 ch^2\chi_{1R}} \\ + 1 - \frac{4\eta^2}{(\eta^2+\xi^2) ch^2\chi_{1R}} + \frac{4\eta^4}{(\eta^2+\xi^2)^2 ch^4\chi_{1R}} \equiv 1. \end{aligned} \tag{2.29}$$

Similarly, using (2.25) and (2.28) we obtain

$$G = \mathbf{r}_y^2 = r_{1y}^2 + r_{2y}^2 + r_{3y}^2 = \frac{4m_{1Ry}^2}{(\eta^2+\xi^2) ch^2\chi_{1R}} \equiv 1, \tag{2.30}$$

$$F = (\mathbf{r}_x, \mathbf{r}_y) = r_{1x}r_{1y} + r_{2x}r_{2y} + r_{3x}r_{3y} = \frac{2\eta m_{1Ry}}{(\eta^2+\xi^2) ch^2\chi_{1R}}. \tag{2.31}$$

Formulas (2.29)–(2.31) give us the first three equations (2.11)–(2.13). Using (2.29)–(2.31) we compute

$$\Lambda = EG - F^2 = \frac{4m_{1Ry}^2(\eta^2 sh^2\chi_{1R} + \xi^2 ch^2\chi_{1R})}{(\eta^2+\xi^2)^2 ch^4\chi_{1R}}. \tag{2.32}$$

We calculate the components of the vector $\mathbf{n}$

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{\sqrt{\Lambda}} = \frac{1}{\sqrt{\Lambda}}(n_1, n_2, n_3), \tag{2.33}$$

$$n_1 = \frac{1}{\sqrt{\Lambda}}(r_{2x}r_{3y} - r_{3x}r_{2y}) = -\frac{2\xi m_{1Ry}}{\sqrt{\Lambda}(\eta^2+\xi^2) ch\chi_{1R}}. \tag{2.34}$$

Similarly, for the components

$$n_2 = \frac{1}{\sqrt{\Lambda}}(r_{3x}r_{1y} - r_{1x}r_{3y}) = -\frac{2\eta sh\chi_{1R} m_{1Ry}}{\sqrt{\Lambda}(\eta^2+\xi^2) ch^2\chi_{1R}}, \tag{2.35}$$

$$n_3 = \frac{1}{\sqrt{\Lambda}}(r_{1x}r_{2y} - r_{2x}r_{1y}) = 0. \tag{2.36}$$

Now, from (2.26), (2.28) we have

$$r_{1xx} = -\frac{2\eta^3 ch\chi_{1R}(ch^2\chi_{1R} - 2sh^2\chi_{1R})}{(\eta^2+\xi^2) ch^4\chi_{1R}} = -\frac{2\eta^3(1 - sh^2\chi_{1R})}{(\eta^2+\xi^2) ch^3\chi_{1R}}, \tag{2.37}$$

$$r_{2xx} = -\frac{2\eta^2\xi sh\chi_{1R}}{(\eta^2+\xi^2) ch^2\chi_{1R}}, \tag{2.38}$$

$$r_{3xx} = \frac{4\eta^3 sh\chi_{1R}}{(\eta^2+\xi^2) ch^3\chi_{1R}}. \tag{2.39}$$

Thus, using (2.34)–(2.39) we can compute

$$r_{3xx} = \frac{4\eta^3 sh\chi_{1R}}{(\eta^2+\xi^2)ch^3\chi_{1R}}. \tag{2.40}$$

Taking into account, that $n_3 = 0$,

$$L = \frac{4\eta^3\xi m_{1Ry}}{\sqrt{\Lambda}(\eta^2+\xi^2)^2 ch^4\chi_{1R}}. \tag{2.41}$$

Similarly, we calculate

$$M = \frac{4\eta^2\xi m_{1Ry}^2}{\sqrt{\Lambda}(\eta^2+\xi^2)^2 ch^4\chi_{1R}}, \tag{2.42}$$

$$N = \frac{4\eta\xi m_{1Ry}^3}{\sqrt{\Lambda}(\eta^2+\xi^2)^2 ch^4\chi_{1R}}. \tag{2.43}$$

The formulas (2.41)–(2.43) give us the last three equations (2.11)–(2.13). Using (2.11)–(2.13), for example, the Gaussian curvature can be calculated

$$\begin{aligned} K &= \frac{LN - M^2}{\Lambda} \\ &= \frac{1}{\Lambda}\left(\frac{4\eta^3\xi m_{1Ry}}{\sqrt{\Lambda}(\eta^2+\xi^2)^2 ch^4\chi_{1R}}\frac{4\eta\xi m_{1Ry}^3}{\sqrt{\Lambda}(\eta^2+\xi^2)^2 ch^4\chi_{1R}} - \frac{16\eta^4\xi^2 m_{1Ry}^4}{\Lambda(\eta^2+\xi^2)^4 ch^8\chi_{1R}}\right) \\ &= \frac{1}{\Lambda}\left(\frac{16\eta^4\xi^2 m_{1Ry}^4}{\Lambda(\eta^2+\xi^2)^4 ch^8\chi_{1R}} - \frac{16\eta^4\xi^2 m_{1Ry}^4}{\Lambda(\eta^2+\xi^2)^4 ch^8\chi_{1R}}\right) \equiv 0. \end{aligned} \tag{2.44}$$

We see that for the surface Gaussian curvature is equal to zero. Theorem is proved. □

3 Conclusion

Based on the results of work [2], where Gauss–Codazzi–Mainardi equation considered in multidimensional space, we have studied equation M-I and built the surface corresponding to soliton solution. Thus, this work fully reveals the meaning of the geometric approach [2] in $(2+1)$-dimensions.

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