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An Optimal Control Problem for a Nonlinear Hyperbolic Equation with an Infinite Time Horizon

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Abstract. An optimization control problem for a nonlinear hyperbolic equation with non-smooth nonlinearity and infinite time horizon without global solvability of the boundary problem is considered. This problem is solved using an approximation. The convergence of the approximation is proved. Necessary conditions of optimality are obtained.

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1. Introduction

We consider an optimization control problem for a system described by a nonlinear hyperbolic equation. The existence and uniqueness of the boundary problem is not guaranteed for arbitrary control data. Furthermore, the nonlinear term of the equation is non-smooth and the problem is considered on an infinite time interval.

Optimization methods for systems described by nonlinear parabolic and elliptic equations are well known. There exist a lot of results for control systems characterized by Goursat–Darboux problems. Some results for optimization problems for usual boundary problems are obtained by Matveev and Yakubovich [1], Tiba [2], Fursikov [3]. They prove the existence of the optimal control and necessary conditions for optimality in the following case. The boundary problem has a unique solution for all admissible control data, nonlinear terms are smooth, and the time interval is finite. Relaxation methods for these problems are used by Tiba [4] and Sumin [5]. Banks and Kunisch [6] apply numerical methods for its solution. Kuliev and Gasanov (see [7]) consider optimization problems for nonlinear hyperbolic equations with a control in coefficients and with state constraints. Optimization methods for systems described by regular boundary problems for nonlinear hyperbolic equations under smoothness assumptions and with finite time horizon are well known.

Singular control systems may be not solvable or may have non-unique solutions for admissible control data. The use of variational methods or gradient methods for functional minimization are a matter of serious difficulty in this situation. The control is the primary object, and the state function is second for standard optimization methods. It is determined by the state equation for the given value of the control. However, the unique solvability of the problem can be violated when varying the control in the singular case. Then the control and the state function should be interpreted as an equal in rights pair. The state equation is interpreted as a constraint in this situation. The cost functional is minimized here on the admissible set of pairs, that is the set of control-state pairs such that the state equation holds true. This conditional extremal problem can be solved by means of the infinite-dimensional Lagrange multipliers method (see Fursikov [3]) or penalty method (see Lions [8]). Optimization problems for nonlinear hyperbolic equations are considered for the singular case in these papers. However the considered systems are smooth and the corresponding time interval is finite.

Two types of non-smooth optimization problems are known. Non-smooth terms can be included either in the cost functional or in the state equation. If the state operator is smooth and there is non-smoothness only in the functional the problem can be solved using nonsmooth analysis methods (see, for example, Rockafellar [9] and Clarke [10]). The classical derivatives (Gataux, Fréchet, some other) can be replaced by its non-smooth extension, for example, sub-gradient or Clarke derivative. Using these methods for problems with non-smooth terms in the equation is very difficult because of the absence of the effective non-smooth analogues of the inverse function theorem and the implicit function theorem. They are used for proving the differentiability of the control-state mapping. However, such optimization problems can be solved by means of smooth approximation of the state equation (see Barbu [11]). This idea is used for nonlinear singular elliptic equation in [12].

The additional difficulty of our optimization problem is the non-compactness of the time interval. Optimization problems with infinite time horizon are well known for systems described by ordinary differential equations (see, for example, Seierstad [13] or Aseev and Kryazhimskiy [14]). The analogous problems for distributed systems are seldom considered. However, we note the result of Lions [15] for systems described by linear parabolic equations. He proposes the approximation of the initial system by the analogous system on the finite time interval. The conditions of optimality for the given problem are obtained after passing to the limit in the necessary conditions of optimality for the approximate problem. But the linearity of the system is used substantially in this case. Optimization problems for nonlinear parabolic equations with infinite time horizon are considered by [16] and Cannarsa and Da Prato [17]. However they solve only feedback problems by means of Hamilton–Jacobi equations.

Optimization problems for nonlinear hyperbolic equations without smoothness and regularity, and with infinite time horizon are not solved yet. We will use some ideas and technical methods for its resolution.

Our system is singular. So we will interpret the state equation as a constraint. The control data and the state function are equal in rights in this situation (see [3] and [8]). The considered problem can be solved using Lagrange multipliers method or a penalty method. These methods are equivalent for the optimization problems of [3] and [8]. However we have additional difficulties because of non-smoothness and non-compactness of the time interval. So we prefer to use the Penalty method because it is an approximation method as opposed to the Lagrange multipliers one.

The peculiarity of our problem is the existence of non-smooth terms in the state equation but not in the cost functional. This difficulty will be overcome by means of smooth approximations of the equation. This idea was used in [11], [18] for regular systems. It will be natural to use two forms of approximation (penalty method and smooth approximation) simultaneously. The approximation of the non-smooth term will be realized in the penalty functional. The corresponding smooth penalty approximation method was used in [12] for an optimization problem for a singular elliptic equation with non-smooth nonlinearity. However, it is not sufficient for obtaining the effective results for the system with infinite time horizon.

We know that an optimization problem for linear parabolic equations with infinite time horizon was solved in [15] by means of the finite time approximation method. The corresponding approximate optimization problem has a finite time interval. We propose to use this idea in our problem. The approximation will be realized here at two stages. At first we will use finite time approximation. The obtained optimization problem will be solved with using of the smooth penalty approximation method.

In known results based on approximation techniques (penalty method [8], smooth approximation method [11], and finite time approximation method [15]) necessary conditions of optimality for the initial problem are obtained by passing to the limit in the optimality conditions for the approximate problems. Here we have several difficulties. The high order of difficulty of the given problem does not allow to obtain an analogous statement. But this peculiarity is not an obstacle for solving the problem.

We will find an approximate solution of the problem, it is necessary to define exactly the notion of approximate solution. It will be best to find an admissible control, which is close enough to the optimal one. However, it is only possible for simple enough optimization problems to obtain this form of approximate solution. There is another notion of approximate solution, which is often used. For weak approximate solutions the aim is to find an admissible control where the value of the minimizing functional is close enough to its minimum on the admissible control set. These two forms of approximate solutions are equivalent for optimization problems well posed in the sense of Tikhonov [19]. But the values of the functionals can be close for controls which are not close if the optimization problem is ill posed.

It is known (see [20]), that the majority of optimization problems are ill posed. So the weak approximate solution is used in the practical solution of optimization problems as a rule.

Unfortunately, it may be hard to find a weak approximate solution if the problem is very difficult. We will define a weaker approximate solution. Both, strong and weak approximate solutions are admissible controls. We determine the weaker approximate solution as a control, which is close enough to some admissible control, and the corresponding value of the minimizing functional, which is close enough to its minimum on the admissible control set. This object is weaker because we permit the realization of the given constraint with some small error and do not require it exactly. So the class of solvable problems is extended by weakening the requirement of the approximate solution. The analogical idea was be realized in [21] for an optimization problem for a singular elliptic equation. Our problem is difficult enough. So we will try to find its weaker approximate solution.

There are different methods of practical solution of optimization problems. The first class includes direct methods. The practical algorithm is determined by the problem statement directly. This is true for example for gradient methods (see [22]).

However, using direct practical methods can be very hard for a difficult problem. The given problem may be transformed to another form (condition of optimality) in this situation and the obtained problem may be easier for using numerical methods. The methods of the second class are realized at two steps, obtaining of the optimality conditions and its immediate resolution. However, using the optimality conditions can be very hard too for very difficult optimization problems. In these situations we can approximate the initial problem. So we obtain a third class of practical optimization methods, which are realized in three steps. At first we approximate the given problem. Then we obtain the conditions of optimality for the approximate problem. The last step is the resolution of optimality conditions for the approximate problem. Our problem is very difficult. So we will use a method of the third class.

2. Statement of the problem

Let Ω be an bounded open n-dimensional set with smooth boundary S and let $Q = \Omega \times (0, \infty)$ and $\Sigma = S \times (0, \infty)$. The state function $y = y(x, t)$ is the solution of the initial boundary value problem

$$y'' - \Delta y + f(y) = v, \qquad (x, t) \in Q, \tag{2.1}$$

$$y = 0, \qquad (x, t) \in \Sigma, \tag{2.2}$$

$$y(x, 0) = \varphi(x), \;\; y'(x, 0) = \psi(x), \qquad x \in \Omega. \tag{2.3}$$

The functions in the right side of the equations (2.3) are known. They satisfy the inclusions

$$\varphi \in H_0^1(\Omega), \;\; \psi \in L_2(\Omega).$$

The function f belongs to the set

$$F = \Big\{ f \in C(\mathbb{R}) : \ \big|f(\eta)\big| \le c\big|\eta\big|^3 \ \ \forall \eta \Big\},$$

where $c > 0$.

The function $v = v(x,t)$ is the control. It is an element of the set

$$V = \Big\{ x \in L_2(Q) : \ \ v(x,t) \in G(x), \ \ (x,t) \in Q \Big\},$$

where $G(x)$ is closed and convex for all x and $0 \in G(x)$. The solution of the boundary problem (2.1)–(2.3) will be found from the space

$$Y = \Big\{ y : \ \ y \in L_\infty\big(0,\infty;\ H_0^1\big), \ \ y' \in L_\infty\big(0,\infty;\ L_2\big) \Big\}.$$

It is important that we cannot guarantee the existence of the solution of this problem for arbitrary control (see [8]). However, we can determine the set U of admissible pairs for the system (2.1)–(2.3) (see [3] and [8]).

Definition 1. *The pair* (v,y) *from the set* $V \times Y$ *is called* **admissible**, *if it satisfies the equations* (2.1)*–*(2.3)*.*

The state functional is determine by

$$I(v,y) = \frac{1}{6}\big\|y - z\big\|^6_{L_6(Q)} + \frac{\alpha}{2}\big\|v\big\|^2_{L_2(Q)},$$

where $\alpha > 0$ and z is a given function from the space $L_6(Q)$. We consider the following optimization problem.

Problem P. *Find an admissible pair* (v,y) *that minimizes the state functional* I *on the set* U*.*

The existence of its solution is guaranteed by the following result.

Theorem 1. *If the set* U *is nonempty, then the Problem* **P** *is solvable.*

Proof. The functional I is bounded from below. Therefore, there exists a minimizing sequence $\{u_n\}$ for this problem. Let $u_n = (v_n, y_n)$, where $v_n \in V$, $\ y_n \in Y$. It satisfies the equations

$$y_n'' - \Delta y_n + f(y_n) = v_n, \qquad (x,t) \in Q, \tag{2.4}$$

$$y_n = 0, \qquad (x,t) \in \Sigma, \tag{2.5}$$

$$y_n(x,0) = \varphi(x), \qquad y_n'(x,0) = \psi(x), \ \ x \in \Omega. \tag{2.6}$$

Furthermore, we have the convergence

$$I(u_n) \to \inf_{u \in U} I(u). \tag{2.7}$$

The sequences $\{v_n\}$ and $\{y_n\}$ are bounded in the spaces $L_2(Q)$ and $L_6(Q)$ because of the coercitivity of the functional. Therefore, the sequence

$$f_n = v_n - f(y_n)$$

is bounded in $L_2(Q)$. The function y_n is the solution of the equation

$$y_n'' - \Delta y_n = f_n, \ \ (x,t) \in Q$$

with boundary conditions (2.5), (2.6). By the classical theory of linear hyperbolic equations (see, for example, [15], Chapter 4) it follows that the sequence $\{y_n\}$ is bounded in the space Y. Choosing a suitable subsequence, we obtain the convergence $v_n \to v$ weakly in $L_2(Q)$ and $y_n \to y$ weakly in Y, in particular $v \in V$. So $y_n \to y$ strongly in $L_2(Q)$ and a.e. in Q by compactness of the embedding from Y into $L_2(Q)$ (see [23], Chapter 1, Theorem 5.1). Thus, $f(y_n) \to f(y)$ a.e. in Q. We get the convergence $f(y_n) \to f(y)$ weakly in $L_2(Q)$ (see [23], Chapter 1, Lemma 1.3). We pass to the limit in the equations (2.4)–(2.6). Then the function y satisfies (2.1)–(2.3). Hence, for $u = (v, y)$. we obtain $u \in U$.

Powers of the norms of the considered spaces are lower semicontinuous. So we get the inequality

$$I(u) \leq \liminf I(u_n).$$

Using (2.7) we obtain, that the pair u is a solution of the Problem **P**. □

Our next step is a solving this problem.

3. Finite time approximation

We will find an approximate solution of the given problem. If the control space is normed, then it is naturally to define an approximate solution as an element of the admissible control set U such that

$$\|u - u_0\| \leq \varepsilon$$

with small enough $\varepsilon > 0$, where u_0 is an exact solution of the given problem.

Unfortunately finding such an approximate solution can be very hard for difficult enough optimization problem. So one often defines a weak approximate solution as a control $u \in U$, that satisfies the inequality

$$\left|I(u) - \inf_{u \in U} I(u)\right| \leq \varepsilon$$

for small enough $\varepsilon > 0$. In reality it is sufficient that

$$I(u) \leq \inf_{u \in U} I(u) + \varepsilon,$$

because the value of the functional in the admissible control cannot be less than its lower bound. The closeness of functional values is a corollary of the closeness of controls if the functional is continuous. Then this approximate solution is weaker than first one. Obviously these notions are equivalent if the optimization problem is well posed in the sense of Tikhonov [19]. But the class of ill-posed optimization problem is much larger. Thus, usually only weak approximate solutions will be found practical optimization problems. Unfortunately, finding a weak approximate solution can be hard too if the optimization problem is very difficult. Then it is necessary to try to determine weaker forms of approximate solutions. Both of the above-defined approximate solutions are elements of the set U. We permit small errors of the optimal control and of the minimum of the functional, but constraints are satisfied exactly. However all objects of the problem statement are only known in approximate form. Therefore it is naturally to require the approximate realization of the given constraints. Of course, the corresponding error should be small. So we have the following weaker form of approximate solutions of optimization problems.

Definition 2. *The control u denotes a* **weaker approximate solution** *of the minimization problem of the functional I on the set U, if $u \in O$ and*

$$I(u) \leq \inf_{u \in U} I(u) + \varepsilon$$

for a small enough neighborhood O of the set U and a small enough value $\varepsilon > 0$.

The weaker approximate solution may be not admissible. But it is close enough to a point of the set U. Besides this, the corresponding value of the functional may exceed its lower bound only by a small value. It is obvious, that the state functional in the weaker approximate solution must be approximated too. If the weaker approximate solution belongs to the set U, it is the weak approximate solution of the given problem.

We have three difficulties of the optimization problem. It is the singularity of the state equation, the smoothness of the nonlinear term, and the non-compactness of the time interval. Therefore, we will use finite time approximation [15], the penalty method [8], and a smooth approximation [16] for finding the approximate solution of the given problem.

Our first step is the finite time approximation. We fix the value $T > 0$. Let $Q_T = \Omega \times (0, T)$ and denote by V_T and Y_T the set of restrictions of functions from V and Y to Q_T. The set U_T of pairs (v, y) is determined from the product $V_T \times Y_T$ such that (2.1) is satisfied on the set Q_T and the initial conditions (2.3) hold true. It is obvious that the restriction of an admissible pair of the system (2.1)–(2.3) to the set Q_T is an element of the set U_T. Furthermore, for all pair $u = (v, y)$ from U_T its trivial extension $\overline{u} = (\overline{v}, \overline{y})$ by zero outside of the set U_T is an admissible pair of the system (2.1)–(2.3). We define the functional

$$I_T(v, y) = \frac{1}{6} \|y - z\|^6_{L_6(Q_T)} + \frac{\alpha}{2} \|v\|^2_{L_2(Q_T)}$$

and consider the finite time approximation problem of the given optimization problem.

Problem $\mathbf{P}_T$. *Find the control data from U_T that minimize the functional I_T on this set.*

Theorem 2. *The problem $\mathbf{P}_T$ is solvable.*

Proof. By lower boundedness of the functional I_T there exists a sequence $\{u_n\}$ from U_T such that

$$I(u_n) \to \inf_{u \in U} I(u). \tag{3.1}$$

We denote $u_n = (v_n, y_n)$. Then we get the equations

$$y_n'' - \Delta y_n + f(y_n) = v_n, \qquad (x,t) \in Q_T, \tag{3.2}$$

$$y_n = 0, \quad x \in S, \qquad t \in (0,T), \tag{3.3}$$

$$y_n(x,0) = \varphi(x), \quad y_n'(x,0) = \psi(x), \qquad x \in \Omega \tag{3.4}$$

from the definition of the set U_T. The sequence $\{y_n\}$ is bounded in the space $L_6(Q_T)$ and $\{v_n\}$ is bounded in $L_2(Q_T)$ because of the coercitivity of the functional I_T. Therefore, the sequence is bounded in the space $L_2(Q_T)$ by definition of the set F. It is obvious that the function y_n is a solution of the equation

$$y_n'' - \Delta y_n = f(y_n), \qquad (x,t) \in Q_T,$$

where

$$f_n = v_n - f(y_n).$$

The sequence $\{f_n\}$ is also bounded in the space $L_2(Q_T)$ and the sequence $\{y_n\}$ is bounded in Y_T by standard theory of linear hyperbolic equations. After passing to subsequences we get $v_n \to v$ weakly in $L_2(Q_T)$, and $y_n \to y$ weakly in Y_T together with $v \in V_T$. We repeat reasoning from the proof of Theorem 1 to obtain $f(y_n) \to f(y)$ weakly in $L_2(Q_T)$. After passing to the limit in the equation (3.2)–(3.4), we obtain that the function y is a solution to the equation (2.1) within Q_T. Thus the pair $u = (v, y)$ belongs to the set U_T. Thus

$$I_T(u) \le \lim\inf I_T(u_n)$$

and the pair u is a solution of the problem $\mathbf{P}_T$ because of (3.1). □

We denote the solution of the approximate problem $\mathbf{P}_T$ by $u_T = (v_T, y_T)$. Now we prove the convergence of the approximation scheme.

Theorem 3. *If* $T \to \infty$ *then* $I(\overline{u}_T) \to \min_{u \in U} I(u)$.

Proof. By the lower boundedness of the functional I on the set U there exists for all $\delta > 0$ a pair u^δ such that

$$I(u^\delta) \le \min_{u \in U} I(u) + \delta.$$

Therefore, we get the inequality

$$I_T(u_T) \le I_T(u^\delta) \le I(u^\delta) \le \min_{u \in U} I(u) + \delta. \tag{3.5}$$

Besides this, we obtain

$$\min_{u\in U} I(u) \le I(\overline{u}_T) = \frac{\alpha}{2}\int\limits_Q \left|\overline{v}_T\right|^2 dQ + \frac{1}{6}\int\limits_Q \left|\overline{y}_T - z\right|^6 dQ$$

$$= \frac{\alpha}{2}\int\limits_{Q_T} \left|\overline{v}_T\right|^2 dQ_T + \frac{1}{6}\int\limits_{Q_T} \left|\overline{y}_T - z\right|^6 dQ_T + \int\limits_T^\infty \int\limits_\Omega \left|z\right|^6 d\Omega dt$$

$$= I_T(u_T) + \int\limits_T^\infty \int\limits_\Omega \left|z\right|^6 d\Omega dt.$$

By (3.5) we get

$$\min_{u\in U} I(u) \le I(\overline{u}_T) \le \min_{u\in U} I(u) + \delta + \int\limits_T^\infty \int\limits_\Omega \left|z\right|^6 d\Omega dt.$$

After the passing to the limit we obtain

$$\min_{u\in U} I(u) \le \lim_{T\to\infty} I(\overline{u}_T) \le \min_{u\in U} I(u) + \delta.$$

Then $I(\overline{u}_T) \to \min_{u\in U} I(u)$ because of arbitrariness of δ. $\square$

By the proved assertion the extension $\overline{u}_T$ of the solution u_T of problem $\mathbf{P}_T$ can be chosen as the weak solution of the initial optimization problem for a small enough value T. It is an admissible pair with the value of the minimizing functional close enough to its minimum. Our next step is an analysis of the problem $\mathbf{P}_T$.

4. Smooth penalty approximation

We use the penalty method with a smooth approximation for solving $\mathbf{P}_T$. For this we define the functional

$$I_T^k(v,y) = \frac{1}{6}\left\|y - z\right\|^6_{L_6(Q_T)} + \frac{\alpha}{2}\left\|v\right\|^2_{L_2(Q_T)} + \frac{1}{2\varepsilon^k}\left\|y'' - \Delta y + f^k(y) - v\right\|^2_{L_2(Q_T)},$$

where $\varepsilon^k > 0$ and $\varepsilon^k \to 0$ as $k \to \infty$ and where f^k are continuous functions with

$$\left(\varepsilon^k\right)^{-1/2}\left\|f^k(y) - f(y)\right\|_{L_2(Q_T)} \to 0 \tag{4.1}$$

uniformly with respect to $y \in Y_T$. We denote further by W_T the set of pairs (v, y) from $(V_T \times Y_T)$ such that the function y satisfies the initial conditions (2.3).

Problem $\mathbf{P}_T^k$ *Find the pair that minimizes the functional I_T^k over the set W_T.*

Theorem 4. *The problem $\mathbf{P}_T^k$ is solvable.*

Proof. Obviously, there exists a sequence $\{u_n\}$ from W_T, where $u_n = (v_n, y_n)$, such that

$$I_T^k(u_n) \to \inf_{u \in U} I_T^k(u). \tag{4.2}$$

By coercitivity of the functional I_k the sequence $\{y_n\}$ is bounded in the space $L_6(Q_T)$ and the sequences $\{v_n\}$ and $\{g_n\}$ are bounded in $L_2(Q_T)$, where

$$g_n = y_n'' - \Delta y_n + f^k(y_n) - v_n.$$

Furthermore, y_n is the solution of the equation

$$y_n'' - \Delta y_n = h_n, \ \ (x,t) \in Q_T$$

with boundary conditions

$$y_n = 0, \ \ x \in S, \ \ t \in (0,T),$$

$$y_n(x,0) = \varphi(x), \ \ y_n'(x,0) = \psi(x), \ \ x \in \Omega,$$

where

$$h_n = g_n + v_n - f^k(y_n).$$

The sequence $\{h_n\}$ is bounded in the space $L_2(Q_T)$. So the sequence $\{y_n\}$ is bounded in the space Y_T. After passing to subsequences we get $v_n \to v$ weakly in $L_2(Q_T)$, $g_n \to g$ weakly in $L_2(Q_T)$, and $y_n \to y$ weakly in Y_T, besides $v \in V_T$. Thus the pair $u = (v, y)$ is an element of U_T. We repeat the reasoning from the proof of Theorem 1 and obtain $f^k(y_n) \to f^k(y)$ weakly in $L_2(Q_T)$. Then $h_n \to h$, where

$$h = g + v - f^k(y).$$

Hence we get

$$I_T^k(u) \le \liminf I_T^k(u_n).$$

Therefore, the pair $u = (v, y)$ is a solution of the problem $\mathbf{P}_T^k$. □

Next, we prove the convergence of the approximation scheme. We denote by $u_T^k = (v_T^k, y_T^k)$ the solution of the problem $\mathbf{P}_T^k$.

Theorem 5. *If $k \to \infty$ then*

$$\liminf I_T(u_T^k) \le \min_{u \in U_T} I_T(u),$$

and $u_T^k \to u_T$ weakly in $L_2(Q_T) \times Y_T$ where u_T is the solution of the problem $\mathbf{P}_T$.

Proof. By lower boundedness of the functional I_T on the set U_T there exists for all $\delta > 0$ a pair $u^\delta = (v^\delta, y^\delta)$ from U_T such that

$$I_T(u^\delta) \le \inf_{u \in U_T} I_T(u) + \delta.$$

Using the optimality of the pair u_T^k for the problem $\mathbf{P}_T^k$, we get

$$\begin{aligned} I_T^k(u_T^k) &= \min_{u \in W_T} I_T^k(u) \le I_T^k(u^\delta) \\ &= I_T(u^\delta) + \frac{1}{2\varepsilon^k} \Big\| \big(y^\delta\big)'' - \Delta y^\delta + f^k(y^\delta) - v^\delta \Big\|^2_{L_2(Q_T)} \\ &\le \min_{u \in U_T} I_T(u) + \delta + \frac{1}{2\varepsilon^k} \Big\| f^k(y^\delta) - f(y^\delta) \Big\|^2_{L_2(Q_T)} \end{aligned}$$

because of the previous inequality and the inclusion $y^\delta \in Y_T$. Using (4.1), we obtain after passing to the limit $k \to \infty$

$$\liminf I_T^k(u_T^k) \le \min_{u \in U_T} I_T(u) + \delta.$$

After passing to the limit $\delta \to 0$, we obtain

$$\liminf I_T^k(u_T^k) \le \min_{u \in U_T} I_T(u). \tag{4.3}$$

Then by definition of the functional I_T^k we obtain the boundedness of the sequences $\{v_T^k\}$ and $\{y_T^k\}$ in the spaces $L_2(Q_T)$ and $L_6(Q_T)$, besides y_T^k is a solution of the equation

$$\big(y_T^k\big)'' - \Delta y_T^k + f^k(y_T^k) - v_T^k = \sqrt{\varepsilon^k} g_T^k, \ \ (x.t) \in Q_T, \tag{4.4}$$

where the sequence $\{g_T^k\}$ is bounded in $L_2(Q_T)$. So the sequence $\{f^k(y_T^k)\}$ is bounded in this space. Thus the function y_T^k satisfies the equation

$$\big(y_T^k\big)'' - \Delta y_T^k = h_T^k,$$

where

$$h_T^k = v_T^k + \sqrt{\varepsilon^k} g_T^k - f^k(y_T^k).$$

It is obvious that the sequence $\{h_T^k\}$ is bounded in the space $L_2(Q_T)$. Using the theory of linear hyperbolic equations, we prove the boundedness of the sequences $\{y_T^k\}$ in the space Y_T.

After passing to subsequences we obtain convergence $v_T^k \to v_T$ weakly in $L_2(Q_T)$, $y_T^k \to y_T$ weakly in Y_T , $g_T^k \to g_T$ weakly in $L_2(Q_T)$, and in particular $v_T \in V_T$. Using the standard method, we get $f^k(y_T^k) \to f(y_T)$ weakly in $L_2(Q_T)$. Then we obtain

$$\begin{aligned} &\Bigg| \int\limits_{Q_T} \Big(f^k(y_T^k) - f(y_T)\Big) \lambda dQ_T \Bigg| \\ &\quad \le \Bigg| \int\limits_{Q_T} \Big(f^k(y_T^k) - f(y_T^k)\Big) \lambda dQ_T \Bigg| + \Bigg| \int\limits_{Q_T} \Big(f(y_T^k) - f(y_T)\Big) \lambda dQ_T \Bigg| \\ &\quad \le \sup_{y \in Y} \Big\| f^k(y) - f(y) \Big\|_{L_2(Q_T)} + \Bigg| \int\limits_{Q_T} \Big(f(y_T^k) - f(y_T)\Big) \lambda dQ_T \Bigg| \end{aligned}$$

for all $\lambda \in L_2(Q_T)$.

Using the uniform convergence of the sequence $\{f^k\}$, we get $f^k(y_T^k) \to f(y_T)$ weakly in $L_2(Q_T)$. After the passing to the limit in (4.4), we obtain, that the function y_T satisfies the equation (2.1) in Q_T. Thus we get the inclusion $y_T \in Y_T$. Then $u_T \in U_T$, where $u_T = (v_T, y_T)$. The following inequality

$$I_T\big(u_T^k\big) \le I_T^k\big(u_T^k\big)$$

is true because of the definition of the functional I_T^k. Using 4.3, we get (if $k \to \infty$)

$$\liminf I_T\big(u_T^k\big) \le \liminf I_T^k\big(u_T^k\big) \le \min_{u \in U_T} I_T\big(u\big).$$

Thus, we obtain

$$I_T(u_T) \le \liminf_{k \to \infty} I_T\big(u_T^k\big) \le \min_{u \in U_T} I_T\big(u\big)$$

from the weak lower semicontinuity of the powers of norms. Therefore u_T is a solution of the problem $\mathbf{P}_T$. □

Therefore, a weaker approximate solution can be found from the obtained results.

Theorem 6. *The extension $\overline{u}_T^k$ of the solution u_T^k of the problem $\mathbf{P}_T^k$ is a weaker approximate solution of the problem $\mathbf{P}$ for large enough k and T.*

Proof. Using Theorem 3, we obtain that for all $\varepsilon > 0$ there exists a value T such that the extension $\overline{u}_T$ of the solution u_T of the problem $\mathbf{P}_T$ satisfies the inequality

$$I\big(\overline{u}_T\big) \le \min_{u \in U} I(u) + \varepsilon/2. \tag{4.5}$$

By Theorem 5, we obtain the convergence $u_T^k \to u_T$ weakly in $L_2(Q_T) \times Y_T$. Therefore,

$$\overline{u}_T^k \to \overline{u}_T \ \text{ weakly in } \ L_2(Q_T) \times Y. \tag{4.6}$$

Furthermore,

$$\liminf_{k \to \infty} I_T\big(u_T^k\big) \le \min_{u \in U_T} I_T(u). \tag{4.7}$$

It is obvious, that $I_T(u) = I(\overline{u})$ for all $u \in L_2(Q_T) \times Y_T$. So we get

$$\min_{u \in U_T} I_T(u) = I_T(u_T) = I(\overline{u}_T)) \le \min_{u \in U} I(u) + \varepsilon/2 \tag{4.8}$$

from inequality (4.5). Based on (4.6) and (4.7), we obtain for given $\varepsilon > 0$, corresponding value T and neighbourhood O of U in the sense of the weak topology of the product $L_2(Q_T) \times Y_T$ a sufficiently large number k such that $\overline{u}_T \in O$ and

$$I\big(\overline{u}_T^k\big) = I_T\big(u_T^k\big) \le I_T(U_T) + \varepsilon/2.$$

Using (4.8), we get

$$I\big(\overline{u}_T^k\big) \le I(U) + \varepsilon/2.$$

Hence, the assertions of the theorem are true. □

Thus we can choose the extension $\overline{u}_T^k$ of the solution to the problem $\mathbf{P}_T^k$ as the weaker approximate solution of the initial optimization problem. Our last step is to obtain necessary conditions of optimality for the problem $\mathbf{P}_T^k$.

5. Solving the approximate optimization problem

We consider the problem $\mathbf{P}_T^k$ of minimizing the smooth functional I_T^k on the convex set W_T. This problem can be solved by means of standard methods.

Theorem 7. *The solution $u_T^k = (v_T^k, y_T^k)$ to the problem $\mathbf{P}_T^k$ satisfies the variational inequality*

$$\int\limits_{Q_T} \big(\alpha v_T^k + p_T^k\big)\big(v - v_T^k\big) dQ_T \geq 0 \qquad \forall v \in V_T, \tag{5.1}$$

where p_T^k is the solution of the initial boundary value problem

$$\big(p_T^k\big)'' - \Delta p_T^k + \big(f^k\big)'\big(y_T^k\big) p_T^k = \big(y_T^k - z\big)^5, \qquad (x,t) \in Q_T, \tag{5.2}$$

$$p_T^k = 0, \qquad x \in \Omega, \ \ t \in (0,T), \tag{5.3}$$

$$p_T^k(x,T) = 0, \qquad \big(p_T^k\big)'(x,T) = 0, \ \ x \in \Omega, \tag{5.4}$$

and y_T^k is the solution of the initial boundary value problem

$$\big(y_T^k\big)'' - \Delta y_T^k + f^k\big(y_T^k\big) = v_T^k + \varepsilon^k p_T^k, \qquad (x,t) \in Q_T, \tag{5.5}$$

$$y_T^k = 0, \qquad x \in \Omega, \ \ t \in (0,T), \tag{5.6}$$

$$y_T^k(x,0) = \varphi(x), \ \ \big(y_T^k\big)'(x,0) = \psi(x), \qquad x \in \Omega. \tag{5.7}$$

Proof. By Theorem 3.1 (of [15], Chapter 1) the minimal point u_T^k of the functional I_T^k on the convex set W_T satisfies the variational inequality

$$\Big\langle \big(I_T^k\big)'\big(u_T^k\big), \, u - u_T^k \Big\rangle \geq 0 \qquad \forall u \in W_T$$

where $\langle \lambda, u \rangle$ is the value of the linear continuous functional λ in the point u, and $\big(I_T^k\big)'\big(u_T^k\big)$ is the Gataux derivative of the functional I_T^k in the point u_T^k. Using the definition of the set W_T, we obtain, that the last formula can be transformed to the variational inequality

$$\Big\langle \big(I_{T_v}^k\big)\big(u_T^k\big), \, v - v_T^k \Big\rangle \geq 0 \qquad \forall v \in V_T \tag{5.8}$$

and stationarity condition

$$I_{T_y}^k\big(u_T^k\big) = 0, \tag{5.9}$$

where $I_{T_v}^k\big(u_T^k\big)$ and $I_{T_y}^k\big(u_T^k\big)$ are the partial derivatives of the functional I_T^k in the considered point. Using the definition of the functional I_T^k we find the values of its partial derivatives from the formula

$$I_{T_v}^k\big(u_T^k\big) = \alpha v_T^k - \frac{1}{\varepsilon^k}\Big(\big(y_T^k\big)'' - \Delta y_T^k + f^k\big(y_T^k\big) - v_T^k\Big),$$

such that

$$\Big\langle \big(I^k_{T_y}\big)\big(u^k_T\big),\, h\Big\rangle = \int\limits_{Q_T} \Big(y^k_T - z\Big)^5 dQ_T$$
$$+ \frac{1}{\varepsilon^k} \int\limits_{Q_T} \Big(\big(y^k_T\big)'' - \Delta y^k_T + f^k\big(y^k_T\big) - v^k_T\Big)\Big(h'' - \Delta h + \big(f^k\big)'\big(y^k_T\big)h\Big) dQ_T$$

holds true for all $h \in Y_T$. We define

$$p^k_T = \frac{1}{\varepsilon^k}\Big(\big(y^k_T\big)'' - \Delta y^k_T + f^k\big(y^k_T\big) - v^k_T\Big).$$

Then the function y^k_T satisfies the equation (5.5). The boundary conditions (5.6), (5.7) can be obtained by using of the definition of the set W_T. We find the value of the partial derivative

$$I^k_{T_v}\big(u^k_T\big) = \alpha v^k_T - p^k_{T.}$$

Thus the variational inequality (5.8) is transformed to (5.1). We obtain in an analogous way the partial derivative from the equation

$$\Big\langle \big(I^k_{T_y}\big)\big(u^k_T\big),\, h\Big\rangle = \int\limits_{Q_T} \bigg(\Big(y^k_T - z\Big)^5 h + p^k_T\Big(h'' - \Delta h + \big(f^k\big)'\big(y^k_T\big)h\Big)\bigg) dQ_T$$
$$= \int\limits_{Q_T} \bigg(\big(y^k_T - z\big)^5 + \big(p^k_T\big)'' - \Delta p^k_T + \big(f^k\big)'\big(y^k_T\big)p^k_T\bigg) h\, dQ_T$$
$$+ \int\limits_{\Omega} \Big(p^k_T(x,T)h'(x,T) - \big(p^k_T\big)'(x,T)h(x,T)\Big)dx$$
$$+ \int\limits_0^T \int\limits_S p^k_T \frac{\partial h}{\partial \vec{n}} dS dt \qquad \forall h \in Y_T,$$

where $\vec{n}$ is the outward normal of S. Using (5.9), we obtain

$$\int\limits_{Q_T} \Big(\big(y^k_T - z\big)^5 + \big(p^k_T\big)'' - \Delta p^k_T + \big(f^k\big)'\big(y^k_T\big)p^k_T\Big) h\, dQ_T$$
$$+ \int\limits_S \Big(p^k_T(x,T)h'(x,T) - \big(p^k_T\big)'(x,T)h(x,T)\Big)dx + \int\limits_0^T \int\limits_S p^k_T \frac{\partial h}{\partial \vec{n}} dS dt = 0$$

for all $h \in Y_T$ and the function p^k_T solves the boundary problem (5.2)–(5.4). □

Thus we have obtained a system including the variational inequality (5.1), the state equations (5.5)–(5.7), and the adjoint system (5.2)–(5.4) for solving the problem $\mathbf{P}^k_T$. It can be computed by using standard iterative methods, see [24]. By Theorem 6 the extension of the solution $u^k_T = (v^k_T, y^k_T)$ of the problem $\mathbf{P}^k_T$

can be chosen as the weaker approximate solution of the initial problem for large enough value of k and T. Note, that the equation (5.5) implies that the initial state equation is satisfied approximately but not exactly because of the second term in its right-hand side. Therefore, the pair $\overline{u}_T^k$ is not admissible and we have indeed a weaker approximate solution of the problem $\mathbf{P}$.

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