Solution of the Neumann Problem of Diffraction by a Strip Using the Wiener–Hopf Method: Short-Wave Asymptotic Solutions

Merey Sautbekova and Seil Sautbekov

***Abstract*— Plane wave diffraction by a strip is considered by the Wiener–Hopf method and the simple short-wave asymptotic solutions are obtained. The Neumann boundary value problem is reduced to the solutions of the Fredholm integral equations of the second kind, and the exact solution in the form of the sum of infinite series has been found by the method of successive approximations. With the help of the saddle point method, the integral operators are calculated and the asymptotic solutions of system are found. The outcomes are compared with supra known asymptotic solutions taking into account tertiary diffraction. Quarternary diffraction was solved and the calculation methodology for the subsequent diffractions was shown.**

***Index Terms*— Antenna theory, boundary value problems, closed-form solutions, electromagnetic diffraction, Fourier transforms, integral equations, microwave propagation, physical theory of diffraction, slot antennas.**

I. Introduction

IT IS KNOWN that the classical problem for diffraction of plane wave at a strip is an important canonical problem. To the present time, there are, in essence, only two rigorous analytical methods for solving diffraction problems: Wiener–Hopf (WH) [1], [2] and Riemann–Hilbert method [3]–[5]. The WH method is also known as the factorization method.

It is well-known that both methods provide closed-form solutions in the case of plane and cylindrical semi-infinite waveguides. Similarly, an exact solution of the half-plane diffraction problem was provided by Sommerfeld [6] since the publication of his famous paper. The method and the results of this paper are detailed in the books [7]–[10].

Following simple WH technique for the half-plane, researchers attempted to combine two similar solutions for spaced edges of conducting surface.

A lot of research has been devoted to asymptotic solutions of diffraction problems on a strip (slit). For the slit problem, the series of successive diffractions of the wave on the edges has been studied in [11]. From the physical point of view, such an approach is characterized by relative separation of each edge contribution to the total diffraction field. Connection between the solution in the form of Schwarzschild series and WH technique has also been considered [12]. Numerous iterative and approximate procedures have been developed to solve the related integral equation [13]–[15].

The diffraction problem for a strip has been considered by the WH method and reduced to a system of Fredholm integral equations of the second kind by Jones [9], as well as by Noble [16]. The modified matrix WH equation for diffraction by a slit in a perfectly conducting plane and a parallel complementary strip has been reduced to a pair of coupled Fredholm integral equations of the second kind and was solved subsequently by iterations [17], [18]. Lateral wave contributions and uniform asymptotic expressions for the high-frequency diffraction by a perfectly conducting strip between two media were also obtained [19].

Some first terms of asymptotic expansion for a diffraction field on parameter $1/ka$ have been obtained on the basis of integral equations in works of Millar [20], Westpfahl [21], Lüneburg [22], Kieburts [23], and Stöckel [24], where k is wavenumber and $2a$ is a width of the strip (slit). Using asymptotic formulas for an elliptical cylinder function, Hansen [25] has selected the expressions from a strict solution of a boundary problem in an elliptic frame corresponding to primary boundary waves of Sommerfeld [6]. The approximate solution of an integral equation of the second kind for the current on the plane screen with a slit has been provided by Grinberg [14].

The improvement of the approximate expressions for the scattered field in compact form has been given by means of the physical theory of diffraction in the papers [8], [15]. The obtained formulas are valid at arbitrary angles of incidence and of observation. One of the most important investigations for diffraction from strip geometry has been provided in asymptotic way by Ufimtsev [26], who obtained the solution for a strip in the form of a series in sequential edge waves excited by its different edges. Even though this series is formally convergent, its elements are integrals of rising multiplicity, and this is very inconvenient from computational point of view.

The diffraction problem of a plane wave, perpendicularly impinged on the strip edge, was also considered in closed-form by the WH method, where the boundary value problem was consecutively solved by a reduction to a system of singular boundary integral equations, and then to a system of Fredholm integral equations of the second kind, which was solved, by three different methods, presented in [27]. Its generalization in the case that the plane wave propagates in

Manuscript received September 30, 2016; revised May 9, 2017; accepted June 23, 2017. Date of publication July 24, 2017; date of current version September 1, 2017. *(Corresponding author: Seil Sautbekov.)*

The authors are with L. N. Gumilyov Eurasian National University, Astana 010008, Kazakhstan (e-mail: sautbek@mail.ru).

Digital Object Identifier 10.1109/TAP.2017.2730923

an arbitrary direction has also been carried out, where the boundary value problem was divided into the two independent problems, which are named by Dirichlet and Neumann, and then have been solved separately [28]. However, the analytical outcomes in graphical form were not presented. Therefore, we shall obtain a compact form, and at the same time a close approximation to the exact solution in the form of a series of sequential edge waves [27].

II. Statement of the Problem. Reduction of the Problem to a System of Integral Equations

Let us consider the following Neumann problem [28]:

$$\frac{\partial^2}{\partial y^2}H_x + \frac{\partial^2}{\partial z^2}H_x + k^2 H_x = 0 \quad (k = k_0 \sin\beta) \tag{1}$$

with boundary condition on the strip

$$\frac{\partial}{\partial y}H_x = 0 \quad \text{for } |z| > a,\ y = 0. \tag{2}$$

To complete the formulation of the diffraction problem, and to ensure the uniqueness of its solution, the above wave equation and the boundary conditions are supplemented by the edge (Meixner condition) and Sommerfeld radiation condition for the scattered field. According to the Meixner condition, the component of current density normal to the edge (or H_x) vanishes at the edge as $r^{(1/2)}$ and the field components E_y, E_z vary as $r^{-(1/2)}$, where r is the distance from the edge.

This boundary value problem was treated in [28] by the WH method and the solution was presented in the form

$$\begin{aligned} H_x(y,z) &= \operatorname{sgn} y \int_{-\infty}^{\infty} e^{i(wz+v|y|)}F(w)dw + H_x^0(y,z)\\ H_x^0(y,z) &= B_0 e^{i(yk\sin\theta_0 + zh)}, \quad h = k\cos\theta_0,\ v = \sqrt{k^2 - w^2} \end{aligned} \tag{3}$$

where $B_0 = \text{const}$ is the incident plane wave amplitude.

The integrand function $F(w)$ represents Fourier component of surface currents density localized substantially on the edges of the strip (at the points $z_1 = a$ and $z_2 = -a$, respectively)

$$\begin{aligned} F(w) &= F_1(w) + F_2(w)\\ F_1(w) &= \frac{1}{\sqrt{k+w}}(A_1(w) + B^-(w))e^{-iwa}\\ F_2(w) &= \frac{1}{\sqrt{k-w}}(A_2(w) + B^+(w))e^{iwa} \end{aligned} \tag{4}$$

where the functions

$$A_1(w) = B_0\frac{\sqrt{k+h}}{2\pi}\frac{e^{iha}}{w-h}, \quad A_2(w) = -B_0\frac{\sqrt{k-h}}{2\pi}\frac{e^{-iha}}{w-h}$$

correspond to plane waves, the bound functions

$$\begin{cases} B^+(w) = -\dfrac{1}{2\pi i}\displaystyle\int_{C^-}\frac{e^{-i2au}}{u-w}\sqrt{\frac{k-u}{k+u}}\left(A_1(u) + B^-(u)\right)du\\ B^-(w) = \dfrac{1}{2\pi i}\displaystyle\int_{C^+}\frac{e^{i2au}}{u-w}\sqrt{\frac{k+u}{k-u}}\left(A_2(u) + B^+(u)\right)du \end{cases} \tag{5}$$

correspond to the cylindrical (or conical) waves, reradiated by the strip edges in the course of their repeated reflections. Here C^- and C^+ are integration contours laying parallel at distance $\mp\delta$ ($0 < \delta < Imk$) from the real axis and consisting of an infinitely narrow loop enveloping a point $u = \pm h$ from below or from above, respectively.

It is noted that solution F satisfies the usual Meixner condition $F \sim w^{-(3/2)}$ at $|w| \to \infty$.

Thus, the assigned boundary value problem was reduced to solve a system (5) of the second kind Fredholm integral equations in B^+ and B^-.

III. Solution of the Boundary Value Problem

Note that the system (5) is split into two equations and can be represented in the form

$$B^{\pm}(w) = f^{\pm}(w) + \frac{1}{2\pi i}\int_{C^+} L^{\pm}(w,u)B^{\pm}(u)du \tag{6}$$

where the kernels

$$L^{\pm}(w,u) = e^{i2au}\sqrt{\frac{k+u}{k-u}}\frac{J(\mp w/k) - J(-u/k)}{u \mp w}$$

as well as the functions

$$\begin{aligned} f^+(w) &= (A_1(w) + A_2(w)(J(-1) - J(h/k)))\\ &\qquad \times (J(-w/k) - J(-h/k))\\ f^-(w) &= (A_2(w) + A_1(w)(J(-1) - J(-h/k)))\\ &\qquad \times (J(w/k) - J(h/k)) \end{aligned}$$

are expressed through the special function [28]

$$\begin{aligned} J(w) &= \frac{1}{2\pi i}\int_{C^+}\frac{e^{i2au}}{(u-w)}\sqrt{\frac{k+u}{k-u}}\\ &= -\frac{1}{2i}H_0^{(1)}(2ak) - \sqrt{\frac{k+w}{k-w}}e^{i2aw}\Gamma(ka, w/k). \end{aligned}$$

Here the special function Γ is presented in Appendix A.

The exact solution of the system (5) can be obtained by means of the integral operator

$$\mathbf{P}(w,u) = \frac{1}{4\pi i}\int_{C^{\mp}}\frac{du}{\sqrt{k^2-u^2}}\ g(w,u) \tag{7}$$

$$g(w,u) = 2\frac{\sqrt{(k+w)(k+u)}}{w+u}e^{ia(w+u)} \tag{8}$$

without resorting to calculation the kernel resolvent in (6). Here, the integration contour $C^{\mp}$ can be C^- or C^+ according to the sign of the exponent e^{iau} in (5).

The system (5) can be represented in the form of two independent recursive equations, similarly (6)

$$\begin{aligned} B^+(w) = [&\mathbf{P}(h,-u_1) + \mathbf{P}(-h,u_2)\mathbf{P}(u_2,-u_1) + \mathbf{P}(-u_3,u_2)\\ &\times K^{-1}(u_3,u_2)B^+(u_2)\mathbf{P}(u_2,-u_1)]K(u_1,w) \end{aligned} \tag{9}$$

$$\begin{aligned} B^-(w) = [&\mathbf{P}(-h,u_1) + \mathbf{P}(h,-u_2)\mathbf{P}(-u_2,u_1) + \mathbf{P}(u_3,-u_2)\\ &\times K^{-1}(-u_3,-u_2)B^-(u_2)\mathbf{P}(-u_2,u_1)]\\ &\times K(-u_1,-w) \end{aligned} \tag{10}$$

where

$$K(u,w) = B_0\frac{\sqrt{k-u}}{2\pi i}\frac{e^{-iua}}{u-w}. \tag{11}$$

Therefore, cyclically substituting the right path of (9) back into the B^+ on the right and ordering the subscripts of u_n

($n = 1, 2, \ldots$), since u_n are the integration variables, it is easy to obtain the solution as the sum of the convergent infinite series

$$B^+(w) = (\mathbf{P}(h, -u_1) + \mathbf{P}(-h, u_2)\mathbf{P}(u_2, -u_1) + \ldots)$$
$$K(u_1, w) = \sum_{n=1}^{\infty} \mathbf{P}_n^+ K(u_1, w) \tag{12}$$

where the operators sequence is designated

$$\mathbf{P}_n^+ = \mathbf{P}((-1)^{n-1}h, (-1)^n u_n) \times \prod_{m=n-1}^{1} \mathbf{P}((-1)^{m-1}u_{m+1}, (-1)^m u_m). \tag{13}$$

Here, the product of operators is carried out such as

$$\prod_{m=3}^{1} \hat{f}_m = \hat{f}_3 \hat{f}_2 \hat{f}_1 \quad (m \geqslant 1).$$

Similarly in (12) we find

$$B^-(w) = (\mathbf{P}(-h, u_1) + \mathbf{P}(h, -u_2)\mathbf{P}(-u_2, u_1) + \ldots)$$
$$K(-u_1, -w) = \sum_{n=1}^{\infty} \mathbf{P}_n^- K(-u_1, -w) \tag{14}$$

where

$$\mathbf{P}_n^- = \mathbf{P}((-1)^n h, (-1)^{n-1} u_n) \times \prod_{m=n-1}^{1} \mathbf{P}((-1)^m u_{m+1}, (-1)^{m-1} u_m). \tag{15}$$

Substituting the function (12) and (14) into the expression (4) and (3), we finally obtain the solution of the boundary value problem

$$H_x(z, y) = B_0\big(H_x^1(z, y) + H_x^2(z, y)\big) \tag{16}$$
$$H_x^{(1)}(z, y) = -(\mathbf{P}(h, -w) + \mathbf{P}(-h, u_2)\mathbf{P}(u_2, -w) + \ldots)$$
$$e^{i(wz+v|y|)} = -\sum_{n=1}^{\infty} \mathbf{P}_n^+ e^{i(u_1 z + \sqrt{k^2-u_1^2}|y|)} \tag{17}$$
$$H_x^{(2)}(z, y) = -(\mathbf{P}(-h, w) + \mathbf{P}(h, -u_2)\mathbf{P}(-u_2, w) + \ldots)$$
$$e^{i(wz+v|y|)} = -\sum_{n=1}^{\infty} \mathbf{P}_n^- e^{i(u_1 z + \sqrt{k^2-u_1^2}|y|)}. \tag{18}$$

It should be noted that the convergence of the Neumann series for the operator $\mathbf{P}$ in the expressions (17) and (18) is satisfied automatically since the norm of the operator is less than unity in the space $\mathbb{C}[-\pi, \pi]$ (see Appendix B).

The physical meaning of the terms in (17) and (18) is that each n-member of the series beginning in the order of increase corresponds to n-diffraction of incident plane wave by edges 1 and 2 (at the points $z_1 = a$, $z_2 = -a$, respectively).

IV. Asymptotic Behavior of the Solutions

It is appropriate to find asymptotic behavior of the integrals in (16) using the saddle point method and introducing the new integration variables α and θ in the polar coordinates

$$w = k \sin\alpha, \quad u = k \sin\beta$$
$$y = r \sin\theta, \quad z = r \cos\theta.$$

The operator $\mathbf{P}(u, w)$ in the new coordinate system takes the form

$$\mathbf{P}(\alpha, \beta) = \frac{1}{4\pi i} \int_S d\beta \; g(\alpha, \beta) \tag{19}$$
$$g(\alpha, \beta) = \left(\frac{1}{\sin\frac{\alpha+\beta}{2}} + \frac{1}{\cos\frac{\alpha-\beta}{2}} \right) e^{ika(\sin\alpha + \sin\beta)} \tag{20}$$

where S is Sommerfeld's contour $(-\pi/2 + i\infty, \pi/2 - i\infty)$ on the complex plane of the integration variable, which can be deformed for ensuring the convergence of the integral.

We note the following useful properties of the function $g(\alpha, \beta)$, which follow from the definition of the function (20):

$$-g(\alpha + 2\pi, \beta) = g(\beta, \alpha) = g(\alpha, \beta)$$
$$g(\alpha, -\pi/2) = g(3\pi/2, \alpha) = 0.$$

We write the functions $H_x^{(1)}$ and $H_x^{(2)}$ in (17) and (18) in the new coordinates and denote them as

$$H_x^{(1)}(r, \theta) = \sum_{n=1}^{\infty} h_n^{(1)}(r, \theta)$$
$$H_x^{(2)}(r, \theta) = \sum_{n=1}^{\infty} h_n^{(2)}(r, \theta). \tag{21}$$

Now let us consider the primary diffraction, which corresponds to

$$h_1^{(1)} = -\mathbf{P}\left(\theta_0 + \frac{\pi}{2}, \alpha + \pi\right) e^{ikr\sin(\alpha+\theta)}.$$

By writing the operator in the integral form and reducing $h_1^{(1)}$ to the form

$$h_1^{(1)} = -\frac{\sqrt{1+\cos\theta_0}}{2\pi i} e^{iha} \int_S \frac{\sqrt{1-\sin\alpha}}{\cos\theta_0 - \sin\alpha} e^{ikR^- \sin(\alpha+\theta)} d\alpha$$

where

$$R^- = \sqrt{(z-a)^2 + y^2} \cong r - a\cos\theta \quad \text{at } r \gg a$$

and using the saddle point approximation, we obtain the following expression for the primary diffraction in the far-field zone:

$$h_1^{(1)} \cong Ag\left(\theta_0 + \frac{\pi}{2}, \theta - \frac{\pi}{2}\right)$$
$$A = \frac{1}{2\sqrt{2\pi kr}} e^{i(kr+\pi/4)}. \tag{22}$$

Hence, the following useful formula follows:

$$\mathbf{P}(\beta, \alpha + \pi) e^{ikr\sin(\alpha+\theta)} \cong -Ag\left(\beta, \theta - \frac{\pi}{2}\right) \tag{23}$$

for subsequent diffractions.

At this point, let us consider the secondary diffraction. It is easy to obtain using (23)

$$h_2^{(1)} = -\mathbf{P}\left(\theta_0 - \frac{\pi}{2}, \beta\right) \mathbf{P}(\beta, \alpha + \pi) e^{ikr\sin(\alpha+\theta)}$$
$$\cong A\mathbf{P}\left(\theta_0 - \frac{\pi}{2}, \beta\right) g\left(\beta, \theta - \frac{\pi}{2}\right). \tag{24}$$

We calculate the integral in (24) using the special function

$$\Gamma(kl, \cos\beta) = \frac{\sin\beta}{2\pi i} \int_S \frac{e^{i2kl(\cos\alpha - \cos\beta)}}{\cos\alpha - \cos\beta} d\alpha \tag{25}$$

which can be represented as

$$\Gamma(kl, \cos\beta) = -1 + \frac{\beta}{\pi} + \sin\beta \int_0^{kl} H_0^{(1)}(2t) e^{-2it\cos\beta} dt \quad (26)$$

where $H_0^{(1)}$ is the Hankel function of the first kind and zeroth order.

Expanding the integrand in partial fractions and using the representation of the special function (25), the outcome of the secondary diffraction is obtained

$$\begin{aligned} h_2^{(1)}/A &= \mathbf{P}\left(\theta_0 - \frac{\pi}{2}, \beta\right) g\left(\beta, \theta - \frac{\pi}{2}\right) \\ &= \Gamma(ka, \cos\theta)) g\left(\theta + \frac{\pi}{2}, \theta_0 - \frac{\pi}{2}\right) \\ &\quad + \Gamma(ka, \cos\theta_0) g\left(\theta_0 + \frac{\pi}{2}, \theta - \frac{\pi}{2}\right). \end{aligned} \quad (27)$$

Now let us consider the tertiary diffraction in (21)

$$\begin{aligned} h_3^{(1)} &= -\mathbf{P}\left(\theta_0 + \frac{\pi}{2}, \gamma + \pi\right) \mathbf{P}(\gamma + \pi, \beta) \\ &\quad \times \mathbf{P}(\beta, \alpha + \pi) e^{ikr\sin(\alpha+\theta)}. \end{aligned} \quad (28)$$

Taking into account the change of direction of the integration contour during the change of the integration variable $\gamma + \pi \to \gamma$ in (28)

$$\mathbf{P}\left(\theta_0 + \frac{\pi}{2}, \gamma + \pi\right) \mathbf{P}(\gamma + \pi, \beta) = -\mathbf{P}\left(\theta_0 + \frac{\pi}{2}, \gamma\right) \mathbf{P}(\gamma, \beta) \quad (29)$$

and also by replacing $\theta_0 - \pi/2 \to \gamma$ in (24), we easily obtain the action of the first two operators in (28)

$$\begin{aligned} h_3^{(1)} &= -A\mathbf{P}\left(\theta_0 + \frac{\pi}{2}, \gamma\right) \\ &\quad \times \left(\Gamma(ka, \cos\theta) g\left(\theta + \frac{\pi}{2}, \gamma\right) \right. \\ &\quad \left. + \Gamma\big(ka, \cos(\gamma + \pi/2) g\left(\gamma + \pi, \theta - \frac{\pi}{2}\right)\right) \end{aligned}$$

or the same

$$\begin{aligned} h_3^{(1)} &= -A\Gamma(ka, \cos\theta)\mathbf{P}\left(\theta_0 + \frac{\pi}{2}, \gamma\right) g\left(\gamma, \theta + \frac{\pi}{2}\right) \\ &\quad - \frac{A}{4\pi i} \int_S \Gamma\big(ka, \cos(\gamma + \pi/2)\big) g\left(\theta_0 + \frac{\pi}{2}, \gamma\right) \\ &\quad \times g\left(\gamma + \pi, \theta - \frac{\pi}{2}\right) d\gamma. \end{aligned}$$

The second term in the above expression can be neglected, since the integrand function vanishes in the saddle point $\gamma = \pi/2$ due to the properties of the special function (26) $(\Gamma(ka, -1) = 0)$.

Using again the expression (27), we obtain

$$\begin{aligned} h_3^{(1)} &= -A\Gamma(ka, \cos\theta)(\Gamma(ka, -\cos\theta) g(\theta + 3\pi/2, \theta_0 + \pi/2) \\ &\quad + \Gamma(ka, -\cos\theta_0) g(\theta_0 + 3\pi/2, \theta + \pi/2)) \end{aligned}$$

or

$$\begin{aligned} h_3^{(1)} &= A\Gamma(ka, \cos\theta)(\Gamma(ka, -\cos\theta) g(\theta - \pi/2, \theta_0 + \pi/2) \\ &\quad + \Gamma(ka, -\cos\theta_0) g(\theta_0 - \pi/2, \theta + \pi/2)) \end{aligned} \quad (30)$$

by virtue of the properties of the function g.

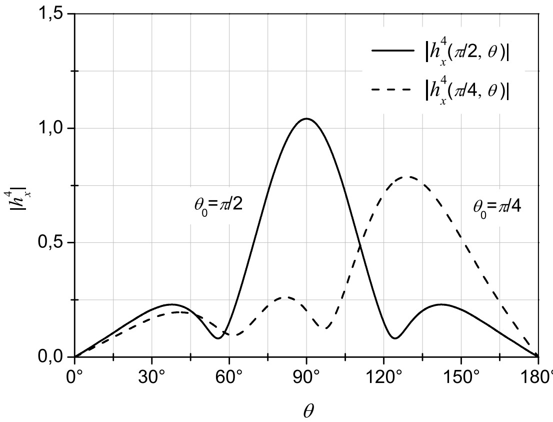


Fig. 1. Directivity patterns of a plane wave diffraction by the strip for the quaternary diffraction.

It should be noted that the expressions $H_x^{(1)}(r, \theta)$ and $H_x^{(2)}(r, \theta)$ in (17) and (18) differ only by opposite signs of the corresponding arguments. Therefore, it is enough to make the following substitutions in (21) for $H_x^{(1)}$:

$$\theta_0 \to \theta_0 - \pi, \quad \theta \to \theta + \pi$$

in order to find an expression for $H_x^{(2)}(r, \theta)$. Finally, by simplifying the expression in (16), we obtain the asymptotic solution of the boundary value problem with only a glance to the tertiary diffraction

$$\begin{aligned} H_x^{(1)}(r, \theta) &= A B_0 T^{(1)}(\theta, \theta_0) g(\theta - \pi/2, \theta_0 + \pi/2) \\ T^{(1)}(\theta, \theta_0) &= (1 + \Gamma(ka, \cos\theta_0)(1 + \Gamma(ka, -\cos\theta)) \\ &\quad + \Gamma(ka, \cos\theta)\Gamma(ka, -\cos\theta) \end{aligned} \quad (31)$$

$$\begin{aligned} H_x^{(2)}(r, \theta) &= A B_0 T^{(2)}(\theta, \theta_0) g(\theta_0 - \pi/2, \theta + \pi/2) \\ T^{(2)}(\theta, \theta_0) &= (1 + \Gamma(ka, \cos\theta))(1 + \Gamma(ka, -\cos\theta_0)) \\ &\quad + \Gamma(ka, \cos\theta)\Gamma(ka, -\cos\theta). \end{aligned} \quad (32)$$

The solution in (31) and (32) contains the outcome of the papers [8], [15] for the tertiary diffraction and clarifies it by the last term

$$\Gamma(ka, \cos\theta)\Gamma(ka, -\cos\theta).$$

Obtained short-wave asymptotic solutions can be easily improved for any order of degree $1/ka$. For example, it is easy to obtain the quarternary diffraction, by extension of the solution in (31) and (32)

$$\begin{aligned} T^{(1)}(\theta, \theta_0) &= (1 + \Gamma(ka, \cos\theta_0)) \\ &\quad \times (1 + \Gamma(ka, -\cos\theta) + \Gamma(ka, \cos\theta) \\ &\quad \times \Gamma(ka, -\cos\theta)) \end{aligned} \quad (33)$$

$$\begin{aligned} T^{(2)}(\theta, \theta_0) &= (1 + \Gamma(ka, -\cos\theta_0)) \\ &\quad \times (1 + \Gamma(ka, \cos\theta + \Gamma(ka, -\cos\theta) \\ &\quad \times \Gamma(ka, \cos\theta)). \end{aligned} \quad (34)$$

The directivity patterns

$$h_x^n(\theta_0, \theta) = -\frac{1}{4Aka} H_x \quad (n = 3, 4)$$

of a plane wave scattered by the strip for nth diffraction are shown in Fig. 1, where the values of the curve parameters

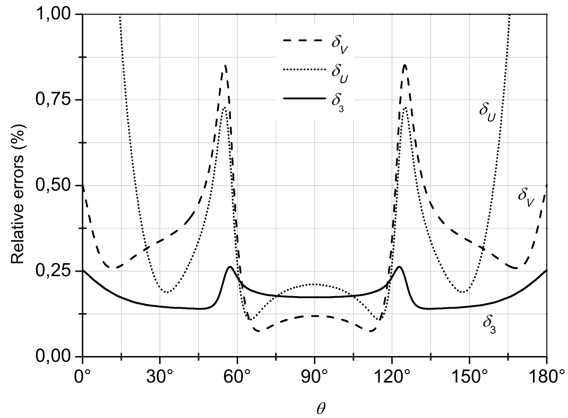


Fig. 2. Relative errors of the asymptotic formulae compared with the quaternary diffraction: Weinstein–Khaskind (δ_V)—dashed line; Ufimtsev (δ_U)—dotted line; and the tertiary diffraction (δ_3)—solid line.

have been chosen as following [26]:

$$B_0 = 1, \quad ka = \sqrt{28}, \; \theta_0 = \frac{\pi}{2} \text{ (solid)}, \; \theta_0 = \pi - \frac{\pi}{4} \text{ (dash)}.$$

V. Conclusion

It is important to observe that, since the WH method is a rigorous method, the additional conditions were not required while solving the problem as well as the obtained solution automatically satisfies the Sommerfeld radiation condition and the boundary Meixner condition on the strip edge which guarantees the uniqueness of the boundary value problem solution.

The exact solution (16) obtained by the WH method in the form of a series of sequential edge waves allows to compare with the asymptotic outcomes. For instance, the tertiary diffraction contains the analytical outcome of Weinstein and Khaskind [8], [15]. Note that the derived (31) and (32) for the tertiary diffraction are compact and easy to use and they also contain all virtues of the solutions obtained by the method of edge waves based on the concept of elementary edge waves. It is easy to obtain the simple asymptotic solution in the form of a quaternary diffraction (34), (35), and so on.

The physical meaning of (17) and (18) obtained by the method of successive approximations consists in multiple diffraction of waves from the strip edges, which is crucial in edge waves method.

Since the directivity patterns of [8], [15], [26] coincide with the corresponding one of Fig. 1 with good graphical accuracy, we therefore introduce the relative errors (relative to h_x^4)

$$\delta_U = \left|\frac{h_x^4 - h}{h}\right|, \quad \delta_V = \left|\frac{h_x^4 - \psi}{\psi}\right|, \; \delta_3 = \left|\frac{h_x^4 - h_x^3}{h_x^3}\right|$$

to compare the outcomes with the functions $h = h(\theta_0, \theta)$ and $\psi = \psi(\theta_0, \theta)$ which have been introduced in [26] and [8], respectively. As it is observed in Fig. 2, the deviation of the tertiary diffraction relative to quaternary is approximately $1/4$ percent (0.25 %). Although the outcomes in [8], [15] do not completely contain tertiary diffraction, the average relative deviation from the tertiary diffraction is the same. It should be noted that the outcomes of [8] and [26], as seen in Fig. 2, differ slightly, except for the grazing directions, in which case the scattered fields also vanish.

In addition to that, their interpretation shows physical completeness of the WH method.

However, it is necessary to point out that the method of successive approximations is not a unique method to solve the system of Fredholm integral equations of the second kind (9), (10).

In the case of the corresponding Dirichlet problem, $g(\alpha, \beta)$ should be substituted by

$$g(\alpha, \beta) = \left(\frac{1}{\sin\frac{\alpha+\beta}{2}} - \frac{1}{\cos\frac{\alpha-\beta}{2}}\right) e^{ika(\sin\alpha + \sin\beta)} \qquad (35)$$

instead of (20).

Appendix A
Special Function Γ

Let us take an advantage of Hankel function representation

$$\pi\, H_0^{(1)}(2bk) = \int_S e^{i2bk\cos\alpha} d\alpha = -\int_{C^+} \frac{e^{i2bu}}{\sqrt{k^2 - u^2}} du \quad \text{(A.1)}$$

where $u = k\cos\alpha$, S is the integration contour ($i\infty - \delta$, $-i\infty + \delta$), $0 < \delta \to 0$, which deviates slightly from the imaginary axis for ensuring the convergence of the integral.

Integrating by the parameter b (from ∞ to l) and multiplying by $\sqrt{k^2 - u^2}$, we have a new representation of special functions

$$\Gamma(kl, w/k) = -\frac{\sqrt{k^2 - w^2}}{2\pi i} \int_{C^+} \frac{e^{i2l(u-w)}}{\sqrt{k^2 - u^2}(u - w)} du$$
$$= \sqrt{k^2 - w^2} \int_{\infty}^{l} H_0^{(1)}(2bk) e^{-i2bw} db \quad (l > 0). \qquad \text{(A.2)}$$

Finally, introducing the new variables $w = k\cos\alpha$, $t = bk$, and by integrating from zero to infinity, we obtain formula (26).

Appendix B
Operator Norm **P**

We estimate the norm of an integral operator $\mathbf{P}$ in $\mathbb{C}[-\pi, \pi]$

$$\|\mathbf{P}\| = \max_{\|f\|=1} \|\mathbf{P}f\| \qquad \text{(B.1)}$$

with help of the inequality

$$\|\mathbf{P}f\| = \max_{\alpha\in[-\pi,\pi]} \left|\frac{1}{4\pi i}\int_S g(\alpha,\beta) f(\beta) d\beta\right|$$
$$\le \max_{\alpha\in[-\pi,\pi]} \left|\frac{1}{4\pi i}\int_S g(\alpha,\beta) d\beta\right| \cdot \max_{\beta\in[-\pi,\pi]} |f(\beta)|$$
$$= \max_{\alpha\in[-\pi,\pi]} \left|\frac{1}{4\pi i}\int_S g(\alpha,\beta) d\beta\right| \qquad \text{(B.2)}$$

where $g \in [-\pi, \pi]^2$, $f \in [-\pi, \pi]$

$$\max_{\beta\in[-\pi,\pi]} |f(\beta)| = \|f\| = 1.$$

In operator theory, a bounded operator $\mathbf{P}$ is said to be a contraction if its operator norm $\|\mathbf{P}\| \leq 1$.

In truth, the above integral

$$I(\alpha) = \frac{1}{4\pi i}\int_S g(\alpha,\beta)d\beta \tag{B.3}$$

is calculated by means of the special function

$$\begin{aligned} X(ka,\beta) &= \frac{1}{4\pi i}\int_S \frac{e^{ika(\cos\alpha-\cos\beta)}}{\sin\frac{\alpha-\beta}{2}}d\alpha \\ &= \frac{e^{-i\pi/4}}{\sqrt{2\pi}}\int_{\infty\sin\frac{\beta}{2}}^{2\sqrt{ka}\sin\frac{\beta}{2}} e^{it^2/2}dt \\ &= -\frac{1}{2}\mathrm{sgn}(\beta)\mathrm{erfc}\left(\mathrm{sgn}(\beta)\sqrt{-2ika}\sin\frac{\beta}{2}\right). \end{aligned} \tag{B.4}$$

Taking into account definition of $g(\alpha,\beta)$ as well as $X(ka,\beta) = -X(ka,-\beta)$ in (B.4) we have

$$\begin{aligned} I(\alpha) &= \frac{1}{4\pi i}\int_S \frac{e^{ika(\cos\beta-\cos(-\alpha-\pi/2))}}{\sin\frac{\beta+\alpha+\pi/2}{2}}d\beta \\ &\quad - \frac{1}{4\pi i}\int_S \frac{e^{ika(\cos\beta-cos(\alpha+\pi/2))}}{\sin\frac{\beta-\alpha-\pi/2}{2}}d\beta \\ &= X(ka,-\alpha-\pi/2) - X(ka,\alpha+\pi/2) \\ &= -2X(ka,\alpha+\pi/2). \end{aligned} \tag{B.5}$$

Since $\max_{\alpha\in[-\pi,\pi]}|X(ka,\alpha)| = (1/2)$ in (B.5) and due to (B.2) it is obvious that

$$\|\mathbf{P}\| \leq \max_{\alpha\in[-\pi,\pi]}|2X(ka,\alpha+\pi/2)| = 1. \tag{B.6}$$

Thus, the operator is contractive, since its norm is less than unity.

REFERENCES

[1] N. Wiener and E. Hopf, "Über eine Klasse singulärer integralgleichungen," Sitzungber. Preuss. Akad. Wiss. Berlin, Phys.-Math., vol. XXXI, pp. 696–706, 1931.

[2] V. A. Fock, "On some integral equations of mathematical physics," *Math. Collection, (in Russian)*, vol. 14, no. 3, pp. 3–50, 1944.

[3] V. P. Shestopalov, *The Method of the Riemann–Hilbert Problem in the Theory of Electromagnetic Wave Diffraction and Propagation*. Vishcha Shkola, Kharkov, 1971.

[4] A. V. Shanin and A. Korolkov, "Diffraction by an impedance strip I. Reducing diffraction problem to Riemann–Hilbert problems," *Quart. J. Mech. Appl. Math.*, vol. 68, no. 3, pp. 321–339, 2015.

[5] A. V. Kisil, "The relationship between a strip Wiener–Hopf problem and a line Riemann–Hilbert problem," *IMA J. Appl. Math.*, vol. 80, no. 5, pp. 1569–1581, 2015.

[6] A. Sommerfeld, "Mathematische theorie der diffraction," *Math. Ann.*, vol. 47, nos. 2–3, pp. 317–374, Jun. 1896.

[7] M. Born and E. Wolf, *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light*. New York, NY, USA: Elsevier, 1980.

[8] L. Weinstein, *The Theory Diffraction Factorization Method*. Boulder, CO, USA: Golem Press, 1969.

[9] D. S. Jones, *Acoustic and Electromagnetic Waves*, vol. 1. New York, NY, USA: Oxford Univ. Press, 1986, p. 764.

[10] H. Hönl, A. Maue, and D. D. K. Westpfahl, *Theorie Der Beugung*. Berlin, Germany: Springer-Verlag, 1961.

[11] K. Schwarzschild, "Die beugung und polarisation des lichts durch einen spalt. I," *Math. Ann.*, vol. 55, no. 2, pp. 177–247, 1901.

[12] A. V. Shanin, "Three theorems concerning diffraction by a strip or a slit," *Quart. J. Mech. Appl. Math.*, vol. 54, no. 1, pp. 107–137, 2001.

[13] S. N. Karp and A. Russek, "Diffraction by a wide slit," *J. Appl. Phys.*, vol. 27, no. 8, pp. 886–894, 1956.

[14] G. A. Greenberg, "A new method for solving the diffraction problem of electromagnetic waves by a plane with a unbounded rectilinear slot, and related it problems," *J. Tech. Phys. (in Russian)*, vol. 27, no. 11, pp. 2595–2605, 1957.

[15] M. Khaskind and L. Weinstein, "Diffraction of plane waves by a slot and strip," *Radioteknika I Elektronika (in Russian)*, vol. 9, no. 10, p. 1800, 1964.

[16] B. Noble, *Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations*. New York, NY, USA: Pergamon Press, 1958.

[17] A. Büyükaksoy and E. Topsakal, "Plane wave diffraction by a slit in a perfectly conducting plane and a parallel complementary strip," *Ann. Telecommun.*, vol. 50, nos. 3–4, pp. 425–432, Mar. 1995.

[18] A. Büyükaksoy and A. Serbest, "Matrix Wiener-Hopf factorization methods and applications to some diffraction problems," in *Analytical and Numerical Methods in Electromagnetic Wave Theory*, M. Hashimoto, M. Idemen, and O. A. Tretyakov, Eds. Tokio, Japan: Science House Co., Ltd., 1993.

[19] S. Sapmaz, K. Kobayashi, A. Büyükaksoy, and G. Uzgören, "High-frequency diffraction by a strip located at the interface between two different media," *Inst. Electron., Inf. Commun. Eng.*, vol. 79, no. 5, pp. 709–719, 1996.

[20] R. Millar, "Diffraction of a wide slit and complementary strip. II," in *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 54, no. 4. Cambridge, U.K.: Cambridge Univ. Press, 1958, pp. 497–511.

[21] K. Westpfahl, "Zur Theorie einer Klasse von Beugungsproblem mittels singulärer integralgleichungen," *Annalen Der Physik*, vol. 459, nos. 6–8, pp. 283–351, 1959. [Online]. Available: http://dx.doi.org/10.1002/andp.19594590602

[22] E. Lüneburg and K. Westpfahl, "Beugung am streifen: Hochfrequenz asymptotik und kleinmansche Lösung," *Ann. Phys.*, vol. 21, pp. 12–25, 1968.

[23] R. B. Kieburtz, "Construction of asymptotic solutions to scattering problems in the Fourier transform representation," *Appl. Sci. Res.*, vol. 12, no. 1, pp. 221–234, Jan. 1965.

[24] H. Stöckel, "Beugung am spalt (II)," *Ann. Phys.*, vol. 471, nos. 5–6, pp. 209–219, 1965.

[25] E. B. Hansen, "Scalar diffraction by an infinite strip and a circular disc," *J. Math. Phys.*, vol. 41, no. 3, pp. 220–245, 1962.

[26] P. Ufimtsev, *Theory of Edge Diffraction in Electromagnetics*. Enrico, CA, USA: Tech Science, 2003.

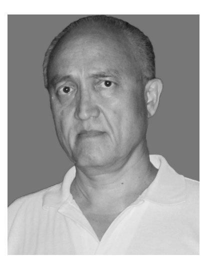
[27] S. Sautbekov, "Factorization method for finite fine structures," *Prog. Electromagn. Res. B*, vol. 25, pp. 1–21, 2010. [Online]. Available: http://mobile.jpier.org/pierb/pier.php?paper=10071801

[28] S. S. Sautbekov, "Diffraction of plane wave by strip with arbitrary orientation of wave vector," *Prog. Electromagn. Res. M*, vol. 21, pp. 117–131, 2011.

Merey Sautbekova received the bachelor's degree in applied mathematics and the master's degree in mathematics from L.N.Gumilyov Eurasian National University, Astana, Kazakhstan, in 2008 and 2010, respectively.

She taught mathematics, mathematics 2, and mathematics 3 during 2010–2011 and 2013–2015.

Dr. Sautbekova received a government grant for Ph.D. in mathematics during 2010-2013.



Seil Sautbekov received the M.S. degree in physics and the Ph.D. degree in physics and mathematics from Al-Farabi Kazakh National University, Almaty, Kazakhstan, in 1977 and 1987, respectively, and the D.Sc. degree in physics and mathematics with a specialization in radiophysics from the Kotel'nikov Institute of Radio Engineering and Electronics, Russian Academy of Sciences, Moscow, Russia, in 2002.

From 1985 to 2004, he held several research and academic positions at Al-Farabi Kazakh National University. Since 2004, he has been a Professor at L.N. Gumilyov Eurasian National University, Astana, Kazakhstan. He has authored over 100 scientific publications.

Dr. Sautbekov was named the Best Lecturer of Higher Education in Kazakhstan in 2007 and was awarded for services in the Development of Sciences of the Republic of Kazakhstan in 2010.