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ELLIPTIC SYSTEMS IN THE PLANE WITH SINGULAR COEFFICIENTS ALONG LINES

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ABSTRACT. The main purpose of present paper consists in investigation of an elliptic systems with Fuchs type operator in the plane. Within the scope of this target investigation we find the solutions of the Dirichlet, Neumann and Robin boundary value problems with the special kind of boundary conditions in the right-hand sides.

Keywords: elliptic systems in the plane, singular coefficients, boundary value problems, angular domain.

AMS Subject Classification: 35J70; 30G20.

1. INTRODUCTION

Let $0 < \varphi_0 \leq 2\pi,\ 0 < \varphi_1 < \varphi_2 < \varphi_0$ and

$$G = \left\{ z = re^{i\varphi} : 0 \leq r < \infty,\ \ 0 \leq \varphi \leq \varphi_0 \right\}.$$

We consider the equation

$$2\overline{z}a_1(\varphi)\partial_{\overline{z}}w + 2za_2(\varphi)\partial_z w + \frac{r^\alpha a_3(\varphi)w}{|y-k_1x|^\alpha} + b(\varphi)\overline{w} = \frac{f(\varphi)r^{\nu+\alpha}}{|y-k_2x|^\alpha} \tag{1}$$

in G, where $a_1(\varphi),\ a_2(\varphi),\ a_3(\varphi),\ b(\varphi),\ f(\varphi) \in C[0,\varphi_0],\ a_1(\varphi) \neq a_2(\varphi)$ for all $\varphi \in [0,\varphi_0]$; $k_1 = \tan\varphi_1$, $k_2 = \tan\varphi_2$, $0 < \alpha < 1$, $\nu > 0$ are real numbers.

Let $p > 1$ if $\nu \geq 1$ and $1 < p < \frac{1}{1-\nu}$ if $\nu < 1$. We will construct the continuous solutions of equation (1) in the class

$$W_p^1(G) \cap C(G). \tag{2}$$

Here $W_p^1(G)$ is the Sobolev space (see [11]). For $\alpha = 0$ and $a_2(\varphi) \equiv 0$ the equation (1) is studied in the articles $[2-10]$.

If we divide the equation (1) by $2\overline{z}a_1(\varphi)$, then it becomes the elliptic equation under $|a_2(\varphi)| < |a_1(\varphi)|$. For $a_2(\varphi) \equiv 0$, the obtained elliptic equation has important application in the theory of infinitesimal bending of surfaces of positive curvature with a point of flattening [7, 9].

The coefficients of this elliptic equation is not belong to $L_q(G),\ q > 2$, therefore it is impossible to apply analytic tools of the theory of general analytic functions, which developed by L. Bers and I. N. Vekua [1, 11]. In order to solve the equation (1) we use a new approach that is described in this article. In present paper we will try to find continuous solutions of the equation (1) as well as solutions of the Dirichlet, Neumann and Robin problems with the special kind of boundary conditions in the right-hand sides.

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2. Construction of continuous solutions to the equation

Using the formulas

$$\frac{\partial}{\partial \bar{z}}=\frac{e^{i\varphi}}{2}\Big(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial \varphi}\Big),\quad \frac{\partial}{\partial z}=\frac{e^{-i\varphi}}{2}\Big(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial \varphi}\Big)$$

the equation (1) can be written in polar coordinates

$$\begin{aligned} r(a_1(\varphi)+a_2(\varphi))\tfrac{\partial w}{\partial r}+i(a_1(\varphi)-a_2(\varphi))\tfrac{\partial w}{\partial \varphi}+\\ +\tfrac{a_3(\varphi)w}{|\sin\varphi-k_1\cos\varphi|^{\alpha}}+b(\varphi)\overline{w}=\tfrac{f(\varphi)r^{\nu}}{|\sin\varphi-k_2\cos\varphi|^{\alpha}}. \end{aligned} \tag{3}$$

We choose solution of the equation (3) from the class (2) in the special form

$$w=r^{\nu}\psi(\varphi), \tag{4}$$

where $\psi=\psi\,(\varphi)$ is a new unknown function from $C^1[0,\varphi_1]$.
Substituting (4) into (3) we obtain

$$\psi'+A(\varphi)\psi=b_1(\varphi)\overline{\psi}+f_1(\varphi), \tag{5}$$

where

$$A(\varphi)=-\frac{i(\nu a_1(\varphi)+\nu a_2(\varphi)+a_3(\varphi))}{|\sin\varphi-k_1\cos\varphi|^{\alpha}\cdot(a_1(\varphi)-a_2(\varphi))},\quad b_1(\varphi)=\frac{ib(\varphi)}{a_1(\varphi)-a_2(\varphi)},$$

$$f_1(\varphi)=-\frac{if(\varphi)}{|\sin\varphi-k_2\cos\varphi|^{\alpha}\cdot(a_1(\varphi)-a_2(\varphi))}.$$

Using the transformation

$$\psi_1=\psi\exp\Big(\int_0^{\varphi}A(\gamma)d\gamma\Big) \tag{6}$$

the equation (5) is transferred into the form

$$\psi_1'(\varphi)=g(\varphi)\overline{\psi_1}+h(\varphi), \tag{7}$$

where

$$g(\varphi)=b_1(\varphi)\exp\Big(-2i\int_0^{\varphi}\Im A(\gamma)d\gamma\Big),\quad h(\varphi)=f_1(\varphi)\exp\Big(\int_0^{\varphi}A(\gamma)d\gamma\Big).$$

Integrating the equation (7) we get

$$\psi_1(\varphi)=\int_0^{\varphi}g(\gamma)\overline{\psi_1(\gamma)}d\gamma+\int_0^{\varphi}h(\gamma)d\gamma+c,$$

where c is any complex number. As $0<\alpha<1$, $a_1(\varphi)\neq a_2(\varphi)$ the integrals of the last equation are convergent. If we introduce

$$(Bf)(\varphi)=\int_0^{\varphi}g(\gamma)\overline{f(\gamma)}d\gamma,\ H(\varphi)=\int_0^{\varphi}h(\gamma)d\gamma$$

then the last equation can be written in the form

$$\psi_1(\varphi)=(B\psi_1)(\varphi)+H(\varphi)+c. \tag{8}$$

For solving the equation (8) we use the iterated scheme

$$(B^0 f)(\varphi) = f(\varphi), \quad (B^n\psi)(\varphi) = (B(B^{n-1}\psi))(\varphi), \quad (n = 1, 2, ...)$$

and the family of functions $\{I_{\nu,n}(\varphi)\}_{n\geq 1}$ that are defined by

$$I_{\nu,1}(\varphi) = \int_0^\varphi g(\gamma)d\gamma, \quad I_{\nu,n}(\varphi) = \int_0^\varphi g(\gamma)\overline{I_{\nu,n-1}(\gamma)}d\gamma. \quad (n = 2, 3, ...)$$

We use the following relations for the further discussions:

$$(Bc)(\varphi) = \bar{c}\, I_{\nu,1}(\varphi), \quad (B(I_{\nu,n}))(\varphi) = I_{\nu,n+1}(\varphi). \quad (n = 2, 3, ...) \tag{9}$$

Applying the operator B to both sides of the equation (8) and using the relations (9) we have get

$$(B\psi_1)(\varphi) = (B^2\psi_1)(\varphi) + (BH)(\varphi) + \bar{c}I_{\nu,1}(\varphi). \tag{10}$$

From (8) and (10) it follows

$$\psi_1(\varphi) = (B^2\psi_1)(\varphi) + (BH)(\varphi) + H(\varphi) + \bar{c}I_{\nu,1}(\varphi) + c. \tag{11}$$

Applying the operator B to both sides of equation (11) and using the relations (9) we have get

$$(B\psi_1)(\varphi) = (B^3\psi_1)(\varphi) + (B^2H)(\varphi) + (BH)(\varphi) + cI_{\nu,2}(\varphi) + \bar{c}I_{\nu,1}(\varphi). \tag{12}$$

From (8) and (12) it follows

$$\psi_1(\varphi) = (B^3\psi_1)(\varphi) + (B^2H)(\varphi) + (BH)(\varphi) + H(\varphi) + \bar{c}I_{\nu,2}(\varphi) + cI_{\nu,1}(\varphi) + c.$$

Continuing this procedure $2n$ times we have

$$\psi_1(\varphi) = (B^{2n+1}\psi_1)(\varphi) + \sum_{j=0}^{2n} (B^jH)(\varphi) + \bar{c}\sum_{j=1}^{n-1} I_{\nu,2j-1}(\varphi) + c\,(1 + \sum_{j=1}^{n} I_{\nu,2j}(\varphi)). \tag{13}$$

Taking into consideration the definition of the operators $(B^j f)(\varphi)$ and the functions $I_{\nu,j}(\varphi)$ the following estimates are obtained:

$$\begin{gathered} \left|(B^j\psi_1)(\varphi)\right| \leq |\psi_1|_0 \tfrac{(|b_1|_0\ \varphi)^j}{j!}, \quad \left|(B^jH)(\varphi)\right| \leq |H|_0\ \tfrac{(|b_1|_0\ \varphi)^j}{j!}, \\ |I_{\nu,j}(\varphi)| \leq \tfrac{(|b_1|_0\ \varphi)^j}{j!},\ (j = 1, 2, ..., n,), \end{gathered} \tag{14}$$

where

$$|f|_0 = \|f\|_{C[0,\varphi_1]}\,.$$

By ensure proceeding to limit $n \to \infty$ in the representation (13), by virtue of the upper estimates (14) we receive

$$\psi_1(\varphi) = \bar{c}P_{\nu,1}(\varphi) + cP_{\nu,2}(\varphi) + F(\varphi), \tag{15}$$

where

$$P_{\nu,1}(\varphi) = \sum_{j=1}^{\infty} I_{\nu,2j-1}(\varphi),\ P_{\nu,2}(\varphi) = 1 + \sum_{j=1}^{\infty} I_{\nu,2j}(\varphi),\ F(\varphi) = \sum_{j=0}^{\infty} (B^jH)(\varphi).$$

Using the estimates (14) and the representation of the functions $P_{\nu,1}(\varphi)$, $P_{\nu,2}(\varphi)$ and $F(\varphi)$ we get

$$|P_{\nu,1}(\varphi)| \leq \sinh\left(|b_1|_0\ \varphi\right), \quad |P_{\nu,2}(\varphi)| \leq \cosh\left(|b_1|_0\ \varphi\right), \quad |F(\varphi)| \leq |h|_0 \exp(|b_1|_0\ \varphi).$$

From (4), (6) and (15) we obtain

$$w\,(r,\varphi) = r^{\nu}\exp\Big(-\int_0^{\varphi} A(\gamma)d\gamma\Big)\left(\bar{c}\,P_{\nu,1}(\varphi) + c\,P_{\nu,2}(\varphi) + F(\varphi)\right). \tag{16}$$

Using the representation of the functions $P_{\nu,1}(\varphi)$, $P_{\nu,2}(\varphi)$ and $F(\varphi)$ we can assert that the function $w\,(r,\varphi)$ given by formula (16) is the solution to the equation (1) belonging to the class (2). Thus we have proved the following Theorem:

Theorem 2.1. *The equation (1) as $a_1(\varphi) \neq a_2(\varphi)$ has an infinitely many solutions belonging the class (2). These solutions are given by (16).*

3. Boundary value problems

Using the representation of the functions $P_{\nu,1}(\varphi)$, $P_{\nu,2}(\varphi)$ and $F(\varphi)$ we receive the following relations:

$$P'_{\nu,2} - g(\varphi)\overline{P_{\nu,1}} = 0, \quad P'_{\nu,1} - g(\varphi)\overline{P_{\nu,2}} = 0, \quad F' - g(\varphi)\overline{F} = h(\varphi).$$

Hence,

$$\begin{gathered} P_{\nu,1}(\varphi) = \textstyle\int_0^{\varphi} g(\gamma)\overline{P_{\nu,2}(\gamma)}d\gamma, \quad P_{\nu,2}(\varphi) = 1 + \int_0^{\varphi} g(\gamma)\overline{P_{\nu,1}(\gamma)}d\gamma, \\ F(\varphi) = \textstyle\int_0^{\varphi} h(\gamma)d\gamma + \int_0^{\varphi} g(\gamma)\overline{F(\gamma)}d\gamma. \end{gathered} \tag{17}$$

From (17) it follows

$$P_{\nu,1}(0) = 0, \quad P_{\nu,2}(0) = 1, \quad F(0) = 0, \tag{18}$$

and

$$P'_{\nu,1}(0) = g(0), \quad P'_{\nu,2}(0) = 0, \quad F'(0) = h(0). \tag{19}$$

Integrating by parts n times the integrals for $P_{\nu,2}(\varphi)$ and $F(\varphi)$ from (17), we get

$$\begin{aligned} P_{\nu,2}\,(\varphi) = 1 + \overline{P_{\nu,1}\,(\varphi)}\sum_{j=1}^{n} I_{\nu,2j-1}\,(\varphi) - P_{\nu,2}\,(\varphi)\sum_{j=0}^{n}\overline{I_{\nu,2j}\,(\varphi)} + \\ + \int_0^{\varphi} I_{\nu,2j}\,(\gamma)g\,(\gamma)\,\overline{P_{\nu,1}\,(\gamma)}d\gamma), \end{aligned}$$

$$\begin{aligned} F(\varphi) = \int_0^{\varphi} h(\gamma)d\gamma + \int_0^{\varphi} h(\gamma)\sum_{j=1}^{n}\overline{I_{\nu,2j}(\gamma)}d\gamma - \int_0^{\varphi}\overline{h(\gamma)}\sum_{j=1}^{n} I_{\nu,2j-1}(\gamma)d\gamma + \\ + \overline{F(\varphi)}\sum_{j=1}^{n} I_{\nu,2j-1}(\varphi) - F(\varphi)\sum_{j=1}^{n}\overline{I_{\nu,2j}(\varphi)} + \int_0^{\varphi} g(\gamma)\overline{F(\gamma)I_{\nu,2n}(\gamma)}d\gamma. \end{aligned}$$

By ensure proceesing to limit $n \to \infty$ in the last relations we have get

$$|P_{\nu,2}(\varphi)|^2 - |P_{\nu,1}(\varphi)|^2 = 1,$$

$$\int_0^{\varphi} h(\gamma)\overline{P_{\nu,2}(\gamma)}d\gamma - \int_0^{\varphi}\overline{h(\gamma)}P_{\nu,1}(\gamma)d\gamma = F(\varphi)\overline{P_{\nu,2}(\varphi)} - \overline{F(\varphi)}P_{\nu,1}(\varphi).$$

Now let us consider the following Dirichlet type boundary value problem.

Problem D. *It is necessary to determine the solution of the equation (1) from the class (2) satisfying the condition*

$$w(r,0) = \beta_1 r^{\nu}, \tag{20}$$

where $\beta_1,\ \nu$ *are given real numbers.*

In order to solve the problem D we will use the formula (16). Substituting the formula (16) in the boundary condition (20) we have get

$$c = \beta_1.$$

Therefore the solution of the problem D is given by

$$w\,(r,\varphi) = r^{\nu}\exp\Big(-\int_0^{\varphi} A(\gamma)d\gamma\Big)(\overline{\beta_1}P_{\nu,1}(\varphi) + \beta_1 P_{\nu,2}(\varphi) + F(\varphi)). \tag{21}$$

Thus we have proved the following Theorem:

Theorem 3.1. *The problem D has a unique solution belonging the class (2). This solution is given by the formula (21).*

Now let us consider the Neumann type boundary value problem.

Problem N. *It is necessary to determine the solution of the equation (1) from the class (2) satisfying the condition*

$$\frac{\partial w}{\partial \varphi}(r,0) = \beta_2 r^{\nu}, \tag{22}$$

where $\beta_2,\ \nu$ *are given real numbers.*

In order to solve the problem D we will again use the formula (16). Substituting the formula (16) in the boundary condition (22) and using (18) and (19) we have get

$$-cA(0) + \overline{c}b_1(0) = \beta_2 - h(0). \tag{23}$$

The first case, let us assume $\delta = |A(0)|^2 - |b_1(0)|^2 \neq 0$. Then the solution of the equation (23) has the form

$$c = \frac{\overline{A(0)}(f_1(0) - \beta_2) + b_1(0)(\overline{h(0)} - \overline{\beta_2})}{\delta}. \tag{24}$$

Now let us consider the second case

$$\delta = 0. \tag{25}$$

In this case the solution to the problem N will be find by the formula

$$c = \begin{cases} \frac{1}{\beta_3}(\Re(\beta_2 - f_1(0)) + i\beta_5(\overline{b_1(0)} - A(0))) & \text{if } \beta_3 \neq 0, \\ \\ \frac{1}{\beta_4}(i\Re(\beta_2 - f_1(0)) - i\beta_6(\overline{b_1(0)} - A(0))) & \text{if } \beta_4 \neq 0, \end{cases} \tag{26}$$

where $\beta_3 = \Re(b_1(0) - A(0)),\ \ \beta_4 = \Im(b_1(0) + A(0));\ \ \ \beta_5,\ \beta_6$ are any real numbers.

Thus we have got the following result.

Theorem 3.2. *a) If $\delta \neq 0$ then the problem N has a unique solution. This solution is given by the formulas (16) and (24); b) If $\delta = 0$ then the problem N has an infinitely many solutions belonging the class (2). These solutions are given by (16) and (26).*

Finally, let us consider the Robin type boundary value problem.

Problem R. *It is necessary to find the solution of the equation (1) from the class (2) satisfying the condition*

$$\frac{\partial w}{\partial \varphi}(r,0) + \gamma w(r,0) = \beta_3 r^{\nu}, \tag{27}$$

where $\gamma,\ \beta_3,\ \nu$ *are given real numbers.*

Substituting the formula (16) in the boundary condition (27) and using (18) and (19) we have get

$$c\gamma + \overline{c}g(0) = \beta_7, \tag{28}$$

where $\beta_7 = \beta_3 + A_\nu(0) - h(0)$.
The first case let us assume $\delta_1 = |\gamma|^2 - |g(0)|^2 \neq 0$. Then the solution of the equation (28) has the form

$$c = \frac{\overline{\beta_7}\gamma + \beta_7\overline{g(0)}}{\delta_1}. \tag{29}$$

Let us consider the second case

$$\delta_1 = 0. \tag{30}$$

In this case the solution of the problem R will be find by the formula

$$c = \begin{cases} \frac{1}{\gamma_1}(\Re(b_1(0)) + i\gamma_2(\overline{g(0)} + \gamma)) & \text{if } \gamma_1 \neq 0, \\ \\ \frac{1}{\gamma_3}(i\Re(b_1(0)) - i\gamma_4(\overline{g(0)} + \gamma)) & \text{if } \gamma_3 \neq 0, \end{cases} \tag{31}$$

where $\gamma_1 = \Re(g(0) + \gamma)$, $\gamma_3 = \Im(g(0) - \gamma)$; γ_2, γ_4 are any real numbers.

Thus we have got the following results.

Theorem 3.3. *a) If $\delta_1 \neq 0$ then the problem R has a unique solution belonging the class (2). This solution is given by the formulas (16) and (29);*

b) If $\delta_1 = 0$ then the problem R has an infinitely many solutions belonging the class (2). These solutions are given by (16) and (31).

4. Conclusion

In present paper some elliptic systems with the Fuchs type operator are investigated. For these systems various kinds of solutions are found. Moreover we have investigated the Dirichlet, Neumann and Robin problems with the special kind of boundary conditions in the right-hand sides.

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