

# Dilatonic dyon-like black hole solutions in the model with two Abelian gauge fields

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**Abstract** Dilatonic black hole dyon-like solutions in the gravitational  $4d$  model with a scalar field, two 2-forms, two dilatonic coupling constants  $\lambda_i \neq 0$ ,  $i = 1, 2$ , obeying  $\lambda_1 \neq -\lambda_2$  and the sign parameter  $\varepsilon = \pm 1$  for scalar field kinetic term are considered. Here  $\varepsilon = -1$  corresponds to a ghost scalar field. These solutions are defined up to solutions of two master equations for two moduli functions, when  $\lambda_i^2 \neq 1/2$  for  $\varepsilon = -1$ . Some physical parameters of the solutions are obtained: gravitational mass, scalar charge, Hawking temperature, black hole area entropy and parametrized post-Newtonian (PPN) parameters  $\beta$  and  $\gamma$ . The PPN parameters do not depend on the couplings  $\lambda_i$  and  $\varepsilon$ . A set of bounds on the gravitational mass and scalar charge are found by using a certain conjecture on the parameters of solutions, when  $1 + 2\lambda_i^2\varepsilon > 0$ ,  $i = 1, 2$ .

## 1 Introduction

In this paper we extend our previous work [1] devoted to dilatonic dyon black hole solutions. We note that at present there exists a certain interest in spherically symmetric solutions, e.g. black hole and black brane ones, related to Lie algebras and Toda chains; see [2–28] and the references therein. These solutions appear in gravitational models with scalar fields and antisymmetric forms.

Here we consider a subclass of dilatonic black hole solutions with electric and magnetic charges  $Q_1$  and  $Q_2$ , respectively, in the  $4d$  model with metric  $g$ , scalar field  $\varphi$ , two 2-forms  $F^{(1)}$  and  $F^{(2)}$ , corresponding to two dilatonic coupling constants  $\lambda_1$  and  $\lambda_2$ , respectively. All fields are defined on an oriented manifold  $\mathcal{M}$ . Here we consider the dyon-like configuration for fields of 2-forms

$$F^{(1)} = Q_1 e^{2\lambda_1\varphi} * \tau, \quad F^{(2)} = Q_2 \tau, \quad (1.1)$$

where  $\tau = \text{vol}[S^2]$  is volume form on 2-dimensional sphere and  $*$  =  $*[g]$  is the Hodge operator corresponding to the oriented manifold  $\mathcal{M}$  with the metric  $g$ . We call this non-composite configuration a dyon-like one in order to distinguish it from the true dyon configuration which is essentially composite and may be chosen in our case either as: (i)  $F^{(1)} = Q_1 e^{2\lambda_1\varphi} * \tau + Q_2 \tau$ ,  $F^{(2)} = 0$ , or (ii)  $F^{(1)} = 0$ ,  $F^{(2)} = Q_1 e^{2\lambda_2\varphi} * \tau + Q_2 \tau$ . From a physical point of view the ansatz (1.1) means that we deal here with a charged black hole, which has two color charges:  $Q_1$  and  $Q_2$ . The charge  $Q_1$  is the electric one corresponding to the form  $F^{(1)}$ , while the charge  $Q_2$  is the magnetic one corresponding to the form  $F^{(2)}$ . For coinciding dilatonic couplings  $\lambda_1 = \lambda_2 = \lambda$  we get a trivial noncomposite generalization of dilatonic dyon black hole solutions in the model with one 2-form which was considered in Ref. [1]; see also [4, 10, 11, 14, 23, 28] and the references therein.

The dilatonic scalar field may be either an ordinary one or a phantom (or ghost) one. The phantom field appears in the action with a kinetic term of the “wrong sign”, which implies the violation of the null energy condition  $p \geq -\rho$ . According to Ref. [29], at the quantum level, such fields could form a “ghost condensate”, which may be responsible for modified gravity laws in the infra-red limit. The observational data do not exclude this possibility [30].

Here we seek relations for the physical parameters of dyonic-like black holes, e.g. bounds on the gravitational mass  $M$  and the scalar charge  $Q_\varphi$ . As in our previous work [1] this problem is solved here up to a conjecture, which states a one-to-one (smooth) correspondence between the pair  $(Q_1^2, Q_2^2)$ , where  $Q_1$  is the electric charge and  $Q_2$  is the magnetic charge, and the pair of positive parameters  $(P_1, P_2)$ , which appear in decomposition of moduli functions at large distances. This conjecture is believed to be valid for all  $\lambda_i \neq 0$  in

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the case of an ordinary scalar field and for  $0 < \lambda_i^2 < 1/2$  for the case of a phantom scalar field (in both cases the inequality  $\lambda_1 \neq -\lambda_2$  is assumed).

### 2 Black hole dyon solutions

Let us consider a model governed by the action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} \left\{ R[g] - \varepsilon g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} e^{2\lambda_1 \varphi} F_{\mu\nu}^{(1)} F^{(1)\mu\nu} - \frac{1}{2} e^{2\lambda_2 \varphi} F_{\mu\nu}^{(2)} F^{(2)\mu\nu} \right\}, \tag{2.1}$$

where  $g = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu$  is the metric,  $\varphi$  is the scalar field,  $F^{(i)} = dA^{(i)} = \frac{1}{2} F_{\mu\nu}^{(i)} dx^\mu \wedge dx^\nu$  is the 2-form with  $A^{(i)} = A_\mu^{(i)} dx^\mu$ ,  $i = 1, 2$ ,  $\varepsilon = \pm 1$ ,  $G$  is the gravitational constant,  $\lambda_1, \lambda_2 \neq 0$  are coupling constants obeying  $\lambda_1 \neq -\lambda_2$  and  $|g| = |\det(g_{\mu\nu})|$ . Here we also put  $\lambda_i^2 \neq 1/2$ ,  $i = 1, 2$ , for  $\varepsilon = -1$ . For  $\lambda_1 = \lambda_2$  the Lagrangian (2.1) appears in the gravitational model with a scalar field and Yang–Mills field with a gauge group of rank 2 (say  $SU(3)$ ) when an Abelian sector of the gauge field is considered.

We consider a family of dyonic-like black hole solutions to the field equations corresponding to the action (2.1), which are defined on the manifold

$$\mathcal{M} = (2\mu, +\infty) \times S^2 \times \mathbb{R}, \tag{2.2}$$

and have the following form:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = H_1^{h_1} H_2^{h_2} \left\{ -H_1^{-2h_1} H_2^{-2h_2} \left( 1 - \frac{2\mu}{R} \right) dt^2 + \frac{dR^2}{1 - \frac{2\mu}{R}} + R^2 d\Omega_2^2 \right\}, \tag{2.3}$$

$$\exp(\varphi) = H_1^{h_1 \lambda_1 \varepsilon} H_2^{-h_2 \lambda_2 \varepsilon}, \tag{2.4}$$

$$F^{(1)} = \frac{Q_1}{R^2} H_1^{-2} H_2^{-A_{12}} dt \wedge dR, \tag{2.5}$$

$$F^{(2)} = Q_2 \tau. \tag{2.6}$$

Here  $Q_1$  and  $Q_2$  are (colored) charges—electric and magnetic, respectively,  $\mu > 0$  is the extremality parameter,  $d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2$  is the canonical metric on the unit sphere  $S^2$  ( $0 < \theta < \pi$ ,  $0 < \phi < 2\pi$ ),  $\tau = \sin \theta d\theta \wedge d\phi$  is the standard volume form on  $S^2$ ,

$$h_i = K_i^{-1}, \quad K_i = \frac{1}{2} + \varepsilon \lambda_i^2, \tag{2.7}$$

$i = 1, 2$ , and

$$A_{12} = (1 - 2\lambda_1 \lambda_2 \varepsilon) h_2. \tag{2.8}$$

The functions  $H_s > 0$  obey the equations

$$R^2 \frac{d}{dR} \left( R^2 \frac{\left( 1 - \frac{2\mu}{R} \right) dH_s}{H_s dR} \right) = -K_s Q_s^2 \prod_{l=1,2} H_l^{-A_{sl}}, \tag{2.9}$$

with the following boundary conditions imposed:

$$H_s \rightarrow H_{s0} > 0 \tag{2.10}$$

for  $R \rightarrow 2\mu$ , and

$$H_s \rightarrow 1 \tag{2.11}$$

for  $R \rightarrow +\infty$ ,  $s = 1, 2$ .

In (2.9) we denote

$$(A_{ss'}) = \begin{pmatrix} 2 & A_{12} \\ A_{21} & 2 \end{pmatrix}, \tag{2.12}$$

where  $A_{12}$  is defined in (2.8) and

$$A_{21} = (1 - 2\lambda_1 \lambda_2 \varepsilon) h_1. \tag{2.13}$$

These solutions may be obtained just by using general formulas for non-extremal (intersecting) black brane solutions from [19–21] (for a review see [22]). The composite analogs of the solutions with one 2-form and  $\lambda_1 = \lambda_2$  were presented in Ref. [1].

The first boundary condition (2.10) guarantees (up to a possible additional requirement on the analyticity of  $H_s(R)$  in the vicinity of  $R = 2\mu$ ) the existence of a (regular) horizon at  $R = 2\mu$  for the metric (2.3). The second condition (2.11) ensures asymptotical (for  $R \rightarrow +\infty$ ) flatness of the metric.

*Remark 1* It should be noted that the main motivation for considering this and more general 4D models governed by the Lagrangian density  $\mathcal{L}$ ,

$$\mathcal{L}/\sqrt{|g|} = R[g] - h_{ab} g^{\mu\nu} \partial_\mu \varphi^a \partial_\nu \varphi^b - \frac{1}{2} \sum_{i=1}^m \exp(2\lambda_{ia} \varphi^a) F_{\mu\nu}^{(i)} F^{(i)\mu\nu}, \tag{2.14}$$

where  $\varphi = (\varphi^a)$  is a set of  $l$  scalar fields,  $F^{(i)} = dA^{(i)}$  are two forms and  $\lambda_i = (\lambda_{ia})$  are dilatonic coupling vectors,  $i = 1, \dots, m$ , is coming from dimensional reduction of supergravity models; in this case the matrix  $(h_{ab})$  is positive definite. For example, one may consider a part of bosonic sector of dimensionally reduced 11d supergravity [16] with  $l$  dilatonic scalar fields and  $m$  2-forms (either originating from the 11d metric or coming from a 4-form) activated; Chern–Simons terms vanish in this case. Certain uplifts (to higher dimensions) of 4d black hole solutions corresponding to (2.14) may lead to black brane solutions in dimensions

$D > 4$ , e.g. to dyonic ones; see [16,17,20,24,25] and the references therein. The dimensional reduction from the 12-dimensional model from Ref. [31] with phantom scalar field and two forms of rank 4 and 5 will lead to the Lagrangian density (2.14) with the matrix  $(h_{ab})$  of pseudo-Euclidean signature.

Equations (2.9) may be rewritten in the following form:

$$\frac{d}{dz} \left[ (1-z) \frac{dy^s}{dz} \right] = -K_s q_s^2 \exp \left( - \sum_{l=1,2} A_{sl} y^l \right), \quad (2.15)$$

$s = 1, 2$ . Here and in the following we use the following notations:  $y^s = \ln H_s$ ,  $z = 2\mu/R$ ,  $q_s = Q_s/(2\mu)$  and  $K_s = h_s^{-1}$  for  $s = 1, 2$ , respectively. We are seeking solutions to Eqs. (2.15) for  $z \in (0, 1)$  obeying

$$y^s(0) = 0, \quad (2.16)$$

$$y^s(1) = y_0^s, \quad (2.17)$$

where  $y_0^s = \ln H_{s0}$  are finite (real) numbers,  $s = 1, 2$ . Here  $z = 0$  (or, more precisely  $z = +0$ ) corresponds to infinity ( $R = +\infty$ ), while  $z = 1$  (or, more rigorously,  $z = 1 - 0$ ) corresponds to the horizon ( $R = 2\mu$ ).

Equations (2.15) with conditions of the finiteness on the horizon (2.17) imposed imply the following integral of motion:

$$\begin{aligned} & \frac{1}{2}(1-z) \sum_{s,l=1,2} h_s A_{sl} \frac{dy^s}{dz} \frac{dy^l}{dz} + \sum_{s=1,2} h_s \frac{dy^s}{dz} \\ & - \sum_{s=1,2} q_s^2 \exp \left( - \sum_{l=1,2} A_{sl} y^l \right) = 0. \end{aligned} \quad (2.18)$$

Equations (2.15) and (2.17) appear for special solutions to the Toda-type equations [20–22]

$$\frac{d^2 z^s}{du^2} = K_s Q_s^2 \exp \left( \sum_{l=1,2} A_{sl} z^l \right), \quad (2.19)$$

for the functions

$$z^s(u) = -y^s - \mu b^s u, \quad (2.20)$$

$s = 1, 2$ , depending on the harmonic radial variable  $u$ :  $\exp(-2\mu u) = 1 - z$ , with the following asymptotical behavior for  $u \rightarrow +\infty$  (on the horizon) imposed:

$$z^s(u) = -\mu b^s u + z_0^s + o(1), \quad (2.21)$$

where  $z_{s0}$  are constants,  $s = 1, 2$ . Here and in the following we denote

$$b^s = 2 \sum_{l=1,2} A^{sl}, \quad (2.22)$$

where the inverse matrix  $(A^{sl}) = (A_{sl})^{-1}$  is well defined due to  $\lambda_1 \neq -\lambda_2$ . This follows from the relations

$$A_{sl} = 2B_{sl}h_l, \quad B_{sl} = \frac{1}{2} + \varepsilon \chi_s \chi_l \lambda_s \lambda_l, \quad (2.23)$$

where  $\chi_1 = +1$ ,  $\chi_2 = -1$  and the invertibility of the matrix  $(B_{sl})$  for  $\lambda_1 \neq -\lambda_2$ , due to the relation  $\det(B_{sl}) = \frac{1}{2}\varepsilon(\lambda_1 + \lambda_2)^2$ .

The energy integral of motion for (2.19), which is compatible with the asymptotic conditions (2.21),

$$\begin{aligned} E &= \frac{1}{4} \sum_{s,l=1,2} h_s A_{sl} \frac{dz^s}{du} \frac{dz^l}{du} \\ & - \frac{1}{2} \sum_{s=1,2} Q_s^2 \exp \left( \sum_{l=1,2} A_{sl} z^l \right) = \frac{1}{2} \mu^2 \sum_{s=1,2} h_s b^s, \end{aligned} \quad (2.24)$$

leads to Eq. (2.18).

*Remark 2* The derivation of the solutions (2.3)–(2.6), (2.9)–(2.11) may be extracted from the relations of [19–21], where the solutions with a horizon were obtained from general spherically symmetric solutions governed by Toda-like equations. These Toda-like equations contain a non-trivial part corresponding to a non-degenerate (quasi-Cartan) matrix  $A$ . In our case these equations are given by (2.19) with the matrix  $A$  from (2.23) and the condition  $\det A \neq 0$  implies  $\lambda_1 \neq -\lambda_2$ . The master equations (2.9) are equivalent to these Toda-like equations. Fortunately, for  $\lambda_1 = -\lambda_2$  and  $\varepsilon = +1$  the solution does exist. It obeys Eqs. (2.3)–(2.6) and (2.9)–(2.11) with  $H_i = H \frac{Q_i^2}{Q_1^2 + Q_2^2}$ ,  $i = 1, 2$ , where  $H = 1 + \frac{P}{R}$  and  $P > 0$  satisfies  $P(P + 2\mu) = K_1(Q_1^2 + Q_2^2)$ ,  $K_1 > 0$ . For  $\lambda_1 = -\lambda_2$  the solution reads

$$\begin{aligned} ds^2 &= H^{h_1} \left\{ -H^{-2h_1} \left( 1 - \frac{2\mu}{R} \right) dt^2 + \frac{dR^2}{1 - \frac{2\mu}{R}} + R^2 d\Omega_2^2 \right\}, \\ \exp(\varphi) &= H^{h_1 \lambda_1 \varepsilon}, \quad F^{(1)} = \frac{Q_1}{R^2} H^{-2} dt \wedge dR, \quad F^{(2)} = Q_2 \tau. \end{aligned}$$

We have verified this solution by using MATHEMATICA. It is also valid for  $\varepsilon = -1$  and  $\lambda_1^2 < \frac{1}{2}$ .

### 3 Some integrable cases

Explicit analytical solutions to Eqs. (2.9), (2.10), (2.11) do not exist. One may try to seek the solutions in the form

$$H_s = 1 + \sum_{k=1}^{\infty} P_s^{(k)} \left(\frac{1}{R}\right)^k, \tag{3.1}$$

where  $P_s^{(k)}$  are constants,  $k = 1, 2, \dots$ , and  $s = 1, 2$ , but only in few integrable cases the chain of equations for  $P_s^{(k)}$  is dropped.

For  $\varepsilon = +1$ , there exist at least four integrable configurations related to the Lie algebras  $A_1 + A_1$ ,  $A_2$ ,  $B_2 = C_2$  and  $G_2$ .

#### 3.1 $(A_1 + A_1)$ -case

Let us consider the case  $\varepsilon = 1$  and

$$(A_{ss'}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}. \tag{3.2}$$

We obtain

$$\lambda_1 \lambda_2 = \frac{1}{2}. \tag{3.3}$$

For  $\lambda_1 = \lambda_2$  we get a dilatonic coupling corresponding to string induced model. The matrix (3.2) is the Cartan matrix for the Lie algebra  $A_1 + A_1$  ( $A_1 = sl(2)$ ). In this case

$$H_s = 1 + \frac{P_s}{R}, \tag{3.4}$$

where

$$P_s(P_s + 2\mu) = K_s Q_s^2, \tag{3.5}$$

$s = 1, 2$ . For positive roots of (3.5)

$$P_s = P_{s+} = -\mu + \sqrt{\mu^2 + K_s Q_s^2}, \tag{3.6}$$

we are led to a well-defined solution for  $R > 2\mu$  with asymptotically flat metric and horizon at  $R = 2\mu$ . We note that in the case  $\lambda_1 = \lambda_2$  the  $(A_1 + A_1)$ -dyon solution has a composite analog which was considered earlier in [7, 10]; see also [15] for certain generalizations.

#### 3.2 $A_2$ -case

Now we put  $\varepsilon = 1$  and

$$(A_{ss'}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{3.7}$$

We get

$$\lambda_1 = \lambda_2 = \lambda, \quad \lambda^2 = 3/2. \tag{3.8}$$

This value of dilatonic coupling constant appears after reduction to four dimensions of the 5d Kaluza–Klein model. We get  $h_s = 1/2$  and (3.7) is the Cartan matrix for the Lie algebra  $A_2 = sl(3)$ . In this case we obtain [20]

$$H_s = 1 + \frac{P_s}{R} + \frac{P_s^{(2)}}{R^2}, \tag{3.9}$$

where

$$2Q_s^2 = \frac{P_s(P_s + 2\mu)(P_s + 4\mu)}{P_1 + P_2 + 4\mu}, \tag{3.10}$$

$$P_s^{(2)} = \frac{P_s(P_s + 2\mu)P_{\bar{s}}}{2(P_1 + P_2 + 4\mu)}, \tag{3.11}$$

$s = 1, 2$  ( $\bar{s} = 2, 1$ ).

In the composite case [1] the Kaluza–Klein uplift to  $D = 5$  gives us the well-known Gibbons–Wilshire solution [5], which follows from the general spherically symmetric dyon solution (related to  $A_2$  Toda chain) from Ref. [4].

#### 3.3 $C_2$ and $G_2$ cases

If we put  $\varepsilon = 1$  and

$$(A_{ss'}) = \begin{pmatrix} 2 & -1 \\ -k & 2 \end{pmatrix} \quad \text{or} \quad (A_{ss'}) = \begin{pmatrix} 2 & -k \\ -1 & 2 \end{pmatrix}, \tag{3.12}$$

we also get integrable configurations for  $k = 2, 3$ , corresponding to the Lie algebras  $B_2 = C_2$  and  $G_2$ , respectively, with the degrees of polynomials (3, 4) and (6, 10). From (2.8), (2.13) and (3.12) we get the following relations for the dilatonic couplings:

$$\frac{1}{2} + \lambda_2^2 = k \left(\frac{1}{2} + \lambda_1^2\right), \quad 1 - 2\lambda_1\lambda_2 = -\frac{1}{2} - \lambda_2^2, \tag{3.13}$$

or

$$\frac{1}{2} + \lambda_1^2 = k \left(\frac{1}{2} + \lambda_2^2\right), \quad 1 - 2\lambda_1\lambda_2 = -\frac{1}{2} - \lambda_1^2. \tag{3.14}$$

Solving Eq. (3.13) we get  $(\lambda_1, \lambda_2) = \pm(\sqrt{2}, \frac{3}{\sqrt{2}})$  for  $k = 2$  and  $(\lambda_1, \lambda_2) = \pm\left(\frac{5}{\sqrt{6}}, 3\sqrt{\frac{3}{2}}\right)$  for  $k = 3$ . The solution to Eq. (3.14) is given by permutation of  $\lambda_1$  and  $\lambda_2$ .

The exact black hole (dyonic-like) solutions for Lie algebras  $B_2 = C_2$  and  $G_2$  will be analyzed in detail in separate publications. They do not exist for the case  $\lambda_1 = \lambda_2$ . We note that for the  $B_2 = C_2$  case ( $k = 2$ ) the polynomials  $H_i$ ,  $i = 1, 2$ , were calculated in [32].

### 3.4 Special solution with two dependent charges

There exists also a special solution

$$H_s = \left(1 + \frac{P}{R}\right)^{b^s}, \tag{3.15}$$

with  $P > 0$  obeying

$$\frac{K_s}{b_s} Q_s^2 = P(P + 2\mu), \tag{3.16}$$

$s = 1, 2$ . Here  $b^s \neq 0$  is defined in (2.22). This solution is a special case of more general ‘‘block orthogonal’’ black brane solutions [33–35].

The calculations give us the following relations:

$$b^s = \frac{2\lambda_{\bar{s}}}{\lambda_1 + \lambda_2} K_s, \tag{3.17}$$

$$Q_s^2 \frac{(\lambda_1 + \lambda_2)}{2\lambda_{\bar{s}}} = P(P + 2\mu) = \frac{1}{2} Q^2, \tag{3.18}$$

where  $s = 1, 2$  and  $\bar{s} = 2, 1$ , respectively. Our solution is well defined if  $\lambda_1 \lambda_2 > 0$ , i.e. the two coupling constants have the same sign.

For positive roots of (3.18)

$$P = P_+ = -\mu + \sqrt{\mu^2 + \frac{1}{2} Q^2} \tag{3.19}$$

we get for  $R > 2\mu$  a well-defined solution with asymptotically flat metric and horizon at  $R = 2\mu$ . It should be noted that this special solution is valid for both signs  $\varepsilon = \pm 1$ . We have

$$ds^2 = H^2 \left\{ -H^{-4} \left(1 - \frac{2\mu}{R}\right) dt^2 + \frac{dR^2}{1 - \frac{2\mu}{R}} + R^2 d\Omega_2^2 \right\}, \tag{3.20}$$

$$\varphi = 0, \tag{3.21}$$

$$F^{(1)} = \frac{Q_1}{H^2 R^2} dt \wedge dR, \quad F^{(2)} = Q_2 \tau, \tag{3.22}$$

where  $H = 1 + \frac{P}{R}$  with  $P$  from (3.19) and

$$Q_1^2 = \frac{\lambda_2}{\lambda_1 + \lambda_2} Q^2, \quad Q_2^2 = \frac{\lambda_1}{\lambda_1 + \lambda_2} Q^2. \tag{3.23}$$

By changing the radial variable,  $r = R + P$ , we get

$$ds^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{3.24}$$

$$F^{(1)} = \frac{Q_1}{r^2} dt \wedge dr, \quad F^{(2)} = Q_2 \tau, \quad \varphi = 0, \tag{3.25}$$

where  $f(r) = 1 - \frac{2GM}{r} + \frac{Q^2}{2r^2}$ ,  $Q^2 = Q_1^2 + Q_2^2$  and  $GM = P + \mu = \sqrt{\mu^2 + \frac{1}{2} Q^2} > \frac{1}{\sqrt{2}} |Q|$ .

The metric in these variables is coinciding with the well-known Reissner–Nordström metric governed by two parameters:  $GM > 0$  and  $Q^2 < 2(GM)^2$ . We have two horizons in this case. Electric and magnetic charges are not independent but obey Eqs. (3.23).

### 3.5 The limiting $A_1$ -cases

In the following we will use two limiting solutions: an electric one with  $Q_1 = Q \neq 0$  and  $Q_2 = 0$ ,

$$H_1 = 1 + \frac{P_1}{R}, \quad H_2 = 1, \tag{3.26}$$

and a magnetic one with  $Q_1 = 0$  and  $Q_2 = Q \neq 0$ ,

$$H_1 = 1, \quad H_2 = 1 + \frac{P_2}{R}. \tag{3.27}$$

In both cases  $P_s = -\mu + \sqrt{\mu^2 + K_s Q^2}$ . These solutions correspond to the Lie algebra  $A_1$ . In various notations the solution (3.26) appeared earlier in [2, 6, 7], and it was extended to the multidimensional case in [6, 7, 12, 13]. The special case with  $\lambda_1^2 = 1/2$ ,  $\varepsilon = 1$ , was considered earlier in [3, 8, 9].

## 4 Physical parameters

Here we consider certain physical parameters corresponding to the solutions under consideration.

### 4.1 Gravitational mass and scalar charge

For ADM gravitational mass we get from (2.3)

$$GM = \mu + \frac{1}{2} (h_1 P_1 + h_2 P_2), \tag{4.1}$$

where the parameters  $P_s = P_s^{(1)}$  appear in Eq. (3.1) and  $G$  is the gravitational constant.

The scalar charge just follows from (2.4):

$$Q_\varphi = \varepsilon (\lambda_1 h_1 P_1 - \lambda_2 h_2 P_2). \tag{4.2}$$

For the special solution (3.15) with  $P > 0$  we get

$$GM = \mu + P = \sqrt{\mu^2 + Q^2}, \quad Q_\varphi = 0. \tag{4.3}$$

For fixed charges  $Q_s$  and the extremality parameter  $\mu$  the mass  $M$  and scalar charge  $Q_\varphi$  are not independent but obey a certain constraint. Indeed, for fixed parameters  $P_s = P_s^{(1)}$  in (3.1) we get

$$y^s = \ln H_s = \frac{P_s}{2\mu} z + O(z^2), \tag{4.4}$$

for  $z \rightarrow +0$ , which after substitution into (2.18) gives (for  $z = 0$ ) the following identity:

$$\frac{1}{2} \sum_{s,l=1,2} h_s A_{sl} P_s P_l + 2\mu \sum_{s=1,2} h_s P_s = \sum_{s=1,2} Q_s^2. \tag{4.5}$$

By using Eqs. (4.1) and (4.2) this identity may be rewritten in the following form:

$$2(GM)^2 + \varepsilon Q_\phi^2 = Q_1^2 + Q_2^2 + 2\mu^2. \tag{4.6}$$

It is remarkable that this formula does not contain  $\lambda$ . We note that in the extremal case  $\mu = +0$  this relation for  $\varepsilon = 1$  was obtained earlier in [14].

### 4.2 The Hawking temperature and entropy

The Hawking temperature corresponding to the solution is found to be

$$T_H = \frac{1}{8\pi\mu} H_{10}^{-h_1} H_{20}^{-h_2}, \tag{4.7}$$

where  $H_{s0}$  are defined in (2.10). Here and in the following we put  $c = \hbar = \kappa = 1$ .

For special solutions (3.15) with  $P > 0$  we get

$$T_H = \frac{1}{8\pi\mu} \left(1 + \frac{P}{2\mu}\right)^{-2}. \tag{4.8}$$

In this case the Hawking temperature  $T_H$  does not depend upon  $\lambda_s$  and  $\varepsilon$ , when  $\mu$  and  $P$  (or  $Q^2$ ) are fixed.

The Bekenstein–Hawking (area) entropy  $S = A/(4G)$ , corresponding to the horizon at  $R = 2\mu$ , where  $A$  is the horizon area, reads

$$S_{BH} = \frac{4\pi\mu^2}{G} H_{10}^{h_1} H_{20}^{h_2}. \tag{4.9}$$

It follows from (4.7) and (4.9) that the product

$$T_H S_{BH} = \frac{\mu}{2G} \tag{4.10}$$

does not depend upon  $\lambda_s$ ,  $\varepsilon$  and the charges  $Q_s$ . This product does not use an explicit form of the moduli functions  $H_s(R)$ .

### 4.3 PPN parameters

Introducing a new radial variable  $\rho$  by the relation  $R = \rho(1 + (\mu/2\rho))^2$  ( $\rho > \mu/2$ ), we obtain the 3-dimensionally conformally flat form of the metric (2.3)

$$g = U \left\{ -U_1 \frac{(1 - (\mu/2\rho))^2}{(1 + (\mu/2\rho))^2} dt \otimes dt + \left(1 + \frac{\mu}{2\rho}\right)^4 \times \delta_{ij} dx^i \otimes dx^j \right\}, \tag{4.11}$$

where  $\rho^2 = |x|^2 = \delta_{ij} x^i x^j$  ( $i, j = 1, 2, 3$ ) and

$$U = \prod_{s=1,2} H_s^{h_s}, \quad U_1 = \prod_{s=1,2} H_s^{-2h_s}. \tag{4.12}$$

The parametrized post-Newtonian (PPN) parameters  $\beta$  and  $\gamma$  are defined by the following standard relations:

$$g_{00} = -(1 - 2V + 2\beta V^2) + O(V^3), \tag{4.13}$$

$$g_{ij} = \delta_{ij}(1 + 2\gamma V) + O(V^2), \tag{4.14}$$

$i, j = 1, 2, 3$ , where  $V = GM/\rho$  is Newton’s potential,  $G$  is the gravitational constant and  $M$  is the gravitational mass (for our case see (4.1)).

The calculations of PPN (or Eddington) parameters for the metric (4.11) give the same result as in [23]:

$$\beta = 1 + \frac{1}{4(GM)^2} (Q_1^2 + Q_2^2), \quad \gamma = 1. \tag{4.15}$$

These parameters do not depend upon  $\lambda_s$  and  $\varepsilon$ . They may be calculated just without knowledge of the explicit relations for the moduli functions  $H_s(R)$ .

These parameters (at least formally) obey the observational restrictions for the solar system [36], when  $Q_s/(2GM)$  are small enough.

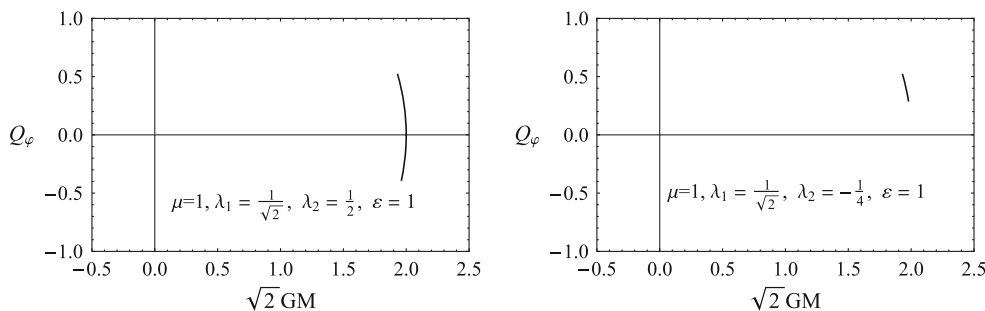
## 5 Bounds on mass and scalar charge

Here we outline the following hypothesis, which is supported by certain numerical calculations [1,37]. For  $h_1 = h_2$  this conjecture was proposed in Ref. [1].

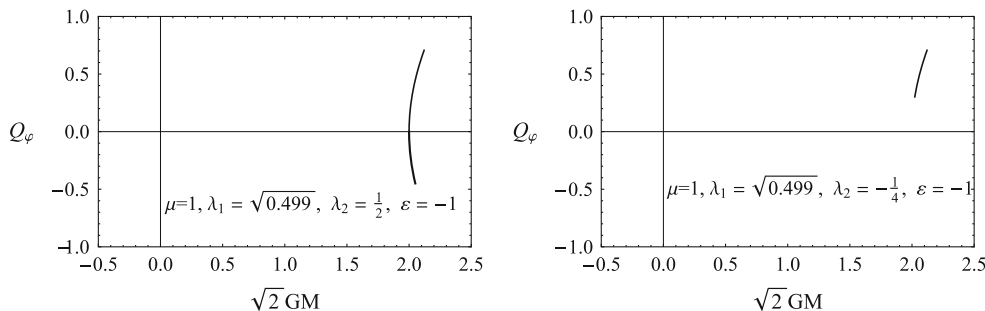
*Conjecture.* For any  $h_1 > 0, h_2 > 0, \varepsilon = \pm 1, Q_1 \neq 0, Q_2 \neq 0$  and  $\mu > 0$ : (A) the moduli functions  $H_s(R)$ , which obey (2.9), (2.10) and (2.11), are uniquely defined and hence the parameters  $P_1, P_2$ , the gravitational mass  $M$  and the scalar charge  $Q_\phi$  are uniquely defined too; (B) the parameters  $P_1, P_2$  are positive and the functions  $P_1 = P_1(Q_1^2, Q_2^2), P_2 = P_2(Q_1^2, Q_2^2)$  define a diffeomorphism of  $\mathbb{R}_+^2$  ( $\mathbb{R}_+ = \{x|x > 0\}$ ); (C) in the limiting case we have: (i) for  $Q_2^2 \rightarrow +0$ :  $P_1 \rightarrow -\mu + \sqrt{\mu^2 + K_1 Q_1^2}, P_2 \rightarrow +0$  and (ii) for  $Q_1^2 \rightarrow +0$ :  $P_1 \rightarrow +0, P_2 \rightarrow -\mu + \sqrt{\mu^2 + K_2 Q_2^2}$ .

The conjecture could be readily verified for the case  $\varepsilon = 1, \lambda_1 \lambda_2 = 1/2$ . Another integrable case  $\varepsilon = 1, \lambda_1 = \lambda_2 = \lambda, \lambda^2 = 3/2$  is more involved [37].

The conjecture implies the following proposition.



**Fig. 1** Graphical illustration of bounds on  $M$  and  $Q_\varphi$  for  $\varepsilon = 1, \lambda_1 = 1/\sqrt{2}, \mu = 1$  and  $Q_1^2 + Q_2^2 = 2$ . The only difference between the two diagrams is  $\lambda_2 = 1/2$  (left panel) and  $\lambda_2 = -1/4$  (right panel)



**Fig. 2** Graphical illustration of bounds on  $M$  and  $Q_\varphi$  for  $\varepsilon = -1, \lambda_1 = \sqrt{0.499}, \mu = 1$  and  $Q_1^2 + Q_2^2 = 2$ . The only difference between the two diagrams is  $\lambda_2 = 1/2$  (left panel) and  $\lambda_2 = -1/4$  (right panel)

**Proposition 1** For  $h_s > 0, Q_s \neq 0, \lambda_s \neq 0$  ( $s = 1, 2$ ) and  $\lambda_1 + \lambda_2 \neq 0$  we have the following bounds on the gravitational mass  $M$  and the scalar charge  $Q_\varphi$ :

$$\begin{aligned} \mu + \frac{h_{\min}}{2} \left( -\mu + \sqrt{h_{\min}^{-1}(Q_1^2 + Q_2^2) + \mu^2} \right) &< GM \\ &\leq \sqrt{\frac{1}{2}(Q_1^2 + Q_2^2) + \mu^2}, \end{aligned} \tag{5.1}$$

$$|Q_\varphi| < |\lambda|_{\max} h_{\min} \left( -\mu + \sqrt{h_{\min}^{-1}(Q_1^2 + Q_2^2) + \mu^2} \right), \tag{5.2}$$

for  $\varepsilon = +1$  ( $0 < h_s < 2$ ) and

$$\begin{aligned} \sqrt{\frac{1}{2}(Q_1^2 + Q_2^2) + \mu^2} &\leq GM < \mu \\ &+ \frac{h_{\max}}{2} \left( -\mu + \sqrt{h_{\max}^{-1}(Q_1^2 + Q_2^2) + \mu^2} \right), \end{aligned} \tag{5.3}$$

$$|Q_\varphi| < |\lambda|_{\max} h_{\max} \left( -\mu + \sqrt{h_{\max}^{-1}(Q_1^2 + Q_2^2) + \mu^2} \right), \tag{5.4}$$

for  $\varepsilon = -1$  ( $h_s > 2$ ). Here  $h_{\min} = \min(h_1, h_2), h_{\max} = \max(h_1, h_2)$ , and  $|\lambda|_{\max} = \max(|\lambda_1|, |\lambda_2|); h_{\min} = (\frac{1}{2} + |\lambda|_{\max}^2)^{-1}$  for  $\varepsilon = +1$  and  $h_{\max} = (\frac{1}{2} - |\lambda|_{\max}^2)^{-1}$  for  $\varepsilon = -1$ .

Here we illustrate the bounds on  $M$  and  $Q_\varphi$  graphically by four figures, which represent a set of physical parameters  $GM$  and  $Q_\varphi$  for  $Q_1^2 + Q_2^2 = Q^2 = 2$  and  $\mu = 1$ .

The left panel of Fig. 1 corresponds to the case  $\varepsilon = +1, \lambda_1 = \sqrt{1/2}$  and  $\lambda_2 = 1/2$ , while the right panel of this figure describes the case  $\varepsilon = +1, \lambda_1 = \sqrt{1/2}$  and  $\lambda_2 = -1/4$ .

On Fig. 2 the left panel illustrates the case  $\varepsilon = -1, \lambda_1 = \sqrt{0.499}$  and  $\lambda_2 = 1/2$ , while the right panel represents the case  $\varepsilon = -1, \lambda_1 = \sqrt{0.499}$  and  $\lambda_2 = -1/4$ .

Two arcs on the left panels of Figs. 1, 2 contain the points with  $Q_\varphi = 0$  corresponding to the special solution from Sect. 3.4.

In proving Proposition 1 we use the following lemma.

**Lemma** Let

$$f(\mu, h; Q^2) = \mu + \frac{h}{2} \left( -\mu + \sqrt{h^{-1}Q^2 + \mu^2} \right), \tag{5.5}$$

be a function of two variables  $\mu > 0$  and  $h > 0$  with fixed value of  $Q^2 > 0$ . Then: (i) for fixed value of  $\mu$  the function  $f(\mu, h; Q^2)$  is monotonically increasing with respect to  $h$ ; (ii) for fixed value of  $h \in (0, 2)$  the function  $f(\mu, h; Q^2)$  is monotonically increasing with respect to  $\mu$  and  $f(+0, h; Q^2) = \frac{1}{2}\sqrt{hQ^2} < f(\mu, h; Q^2)$ .

The proof of the lemma is trivial: item (i) just follows from the identity

$$f(\mu, h; Q^2) = \mu + \frac{Q^2}{2(\mu + \sqrt{h^{-1}Q^2 + \mu^2})}, \tag{5.6}$$

while item (ii) could be readily verified by using the relation

$$\frac{\partial f}{\partial \mu} = 1 + \frac{h}{2} \left( -1 + \frac{\mu}{\sqrt{h^{-1}Q^2 + \mu^2}} \right) > 0 \tag{5.7}$$

for  $h \in (0, 2)$ .

*Proof of Proposition 1* Let us prove Eqs. (5.1), (5.2), (5.3) and (5.4) using the conjecture. The right inequality (or equality) in (5.1) just follows from Eq. (4.6), while the left inequality (or equality) in (5.3) follows from (4.6) and  $M > 0$ , which is valid due to Eq. (4.1),  $h > 0$  and the inequalities  $P_1 > 0$ ,  $P_2 > 0$  (due to the conjecture).

Now let us verify the left inequality in (5.1). We fix the charges by the relation  $Q_1^2 + Q_2^2 = Q^2$ ,  $Q > 0$ , and put  $Q_1^2 = \frac{1}{2}Q^2(1+x)$ ,  $Q_2^2 = \frac{1}{2}Q^2(1-x)$ , where  $-1 < x < 1$ . Due to (4.6) and  $M > 0$  we can use the following parametrization:

$$\sqrt{2}GM = R \cos \psi, \quad Q_\varphi = R \sin \psi, \quad R = \sqrt{Q^2 + 2\mu^2}, \tag{5.8}$$

where  $|\psi| < \pi/2$ . Due to the conjecture and Eqs. (4.1), (4.2) we see that  $\psi = \psi(x)$  is a smooth function which obeys

$$\psi(1-0) = \psi_1, \quad \psi(-1+0) = \psi_2. \tag{5.9}$$

Here  $R \cos \psi_i = \sqrt{2}(\mu + \frac{h_i}{2} P_i)$  and  $R \sin \psi_i = \lambda_i h_i P_i \chi_i$ , where  $P_i = -\mu + \sqrt{K_i Q^2 + \mu^2}$ , ( $K_i = h_i^{-1}$ )  $i = 1, 2$ , and  $\chi_1 = 1$ ,  $\chi_2 = -1$ .

We put  $\lambda_1 > 0$  without loss of generality. The limit  $x \rightarrow +1 - 0$  corresponds to a pure electric black hole while the limit  $x \rightarrow -1 + 0$  corresponds to a pure magnetic one.

To prove Eqs. (5.1) and (5.2) one should verify the inequality

$$\psi_2 < \psi(x) < \psi_1 \tag{5.10}$$

for all  $x \in (-1, 1)$ . Indeed, due to Eqs. (5.10) the points  $(\sqrt{2}GM, Q_\varphi)$  describe an open arc in the circle (see Fig. 1). One of the endpoints of this arc with  $\psi = \psi_{i_0}$ ,  $i_0 = 1, 2$ , gives us the lower bound for  $GM$  and upper bound for  $|Q_\varphi|$ . Due to the lemma this point corresponds to  $i_0$  obeying  $h_{i_0} = h_{\min} = \min(h_1, h_2)$ ,  $P_{i_0} = -\mu + \sqrt{K_{i_0} Q^2 + \mu^2}$  and  $P_{\bar{i}_0} = 0$ .

Let us suppose that (5.10) is not valid. Without loss of generality we put  $\psi(x_*) \geq \psi_1$  for some  $x_*$ . Then, using (5.9) and

the smoothness of the function  $\psi(x)$ , we get for some  $x_1 \neq x_2$ :  $\psi(x_1) = \psi(x_2)$ . This follows from the intermediate value theorem which states that if  $f(x)$  is a continuous function on the interval  $[a, b]$ , then, for any  $d \in [f(a), f(b)]$ , there is a point  $c \in [a, b]$  such that  $f(c) = d$ . (Here for  $f(a) > f(b)$ ,  $[f(a), f(b)]$  is meant to mean  $[f(b), f(a)]$ .) Hence for two different sets  $(Q_1^2, Q_2^2)_1 \neq (Q_1^2, Q_2^2)_2$  we obtain the same coinciding sets:  $(GM, Q_\varphi)_1 = (GM, Q_\varphi)_2$  and hence  $(P_1, P_2)_1 = (P_1, P_2)_2$ ; see (4.1), (4.2) and  $\lambda_1 \neq -\lambda_2$ . But due to our conjecture the map  $(Q_1^2, Q_2^2) \mapsto (P_1, P_2)$  is bijective (i.e. it is one-to-one correspondence). This implies  $(P_1, P_2)_1 \neq (P_1, P_2)_2$ . We get a contradiction which proves our proposition for  $\varepsilon = 1$  and arbitrary  $Q_1^2 + Q_2^2 > 0$ .

The proofs of the right inequality in (5.3) and the bound (5.4) for  $\varepsilon = -1$  are quite similar to that for  $\varepsilon = 1$ . The only difference here is the use of the parametrization

$$\sqrt{2}GM = R \cosh \psi, \quad Q_\varphi = R \sinh \psi, \quad R = \sqrt{Q^2 + 2\mu^2}, \tag{5.11}$$

instead of (5.8). Due to Eqs. (5.10) the points  $(\sqrt{2}GM, Q_\varphi)$  describe an open arc in the hyperbola (see Fig. 2). One of the endpoints of this arc with  $\psi = \psi_{j_0}$ ,  $j_0 = 1, 2$ , gives us the upper bound for  $GM$  and the upper bound for  $|Q_\varphi|$ . Due to the lemma this point corresponds to  $j_0$  obeying  $h_{j_0} = h_{\max} = \max(h_1, h_2)$ ,  $P_{j_0} = -\mu + \sqrt{K_{j_0} Q^2 + \mu^2}$  and  $P_{\bar{j}_0} = 0$ . Thus, Proposition 1 is proved.

Proposition 1 and the lemma imply the following proposition.

**Proposition 2** *In the framework of the conditions of Proposition 1, the following bounds on the mass and scalar charge are valid for all  $\mu > 0$ :*

$$\frac{1}{2} \sqrt{h_{\min}(Q_1^2 + Q_2^2)} < GM, \tag{5.12}$$

$$|Q_\varphi| < |\lambda|_{\max} \sqrt{h_{\min}(Q_1^2 + Q_2^2)}, \tag{5.13}$$

for  $\varepsilon = +1$  ( $0 < h_s < 2$ ), and

$$\sqrt{\frac{1}{2}(Q_1^2 + Q_2^2)} < GM, \tag{5.14}$$

$$|Q_\varphi| < |\lambda|_{\max} \sqrt{h_{\max}(Q_1^2 + Q_2^2)}, \tag{5.15}$$

for  $\varepsilon = -1$  ( $h_s > 2$ ).

In proving (5.13) and (5.15) the following (obvious) relation was used:

$$h(-\mu + \sqrt{h^{-1}Q^2 + \mu^2}) = \frac{Q^2}{\mu + \sqrt{h^{-1}Q^2 + \mu^2}}.$$

In Ref. [1] Propositions 1 and 2 were proved for the case  $\lambda_1 = \lambda_2$  ( $h_1 = h_2$ ). In this case the bound (5.12) is coinciding



(up to notations) with the bound (6.16) from Ref. [11] (BPS-like inequality), which was proved there by using certain spinor techniques.

*Remark 3* When one of  $h_s$ , say  $h_1$ , is negative, the conjecture is not valid. This may be verified just by analyzing the solutions with small enough charge  $Q_2$ .

We note that here we were dealing with a special class of solutions with phantom scalar field ( $\varepsilon = -1$ ). Even in the limiting case  $Q_2 = +0$  and  $Q_1 \neq 0$  there exist phantom black hole solutions which are not covered by our analysis [38] (see also [39]).

*Remark 4* The inequalities on the mass (5.1) and (5.3) in Proposition 1 can be refined when  $\lambda_1\lambda_2 < 0$ . For both cases which are considered in Proposition 1, we get (see right panels of Figs. 1, 2)

$$f(\mu, h_{\min}; Q^2) < GM < f(\mu, h_{\max}; Q^2), \quad (5.16)$$

where  $Q^2 = Q_1^2 + Q_2^2$  and  $f(\mu, h; Q^2)$  is defined in (5.6). The bounds on mass (5.16) are a specific feature of the model with two different dilatonic couplings of opposite sign. For  $\lambda_1\lambda_2 > 0$ , e.g. for  $\lambda_1 = \lambda_2$ , one should use Eqs. (5.1) and (5.3). We also note that in the proof of Proposition 1 the condition  $\lambda_1 \neq -\lambda_2$  was used. For the case  $\lambda_1 = -\lambda_2$  the arcs on the right panels of Figs. 1, 2 reduce to points and we get  $GM = f(\mu, h_1; Q^2)$ .

## 6 Conclusions

In this paper a family of non-extremal black hole dyon-like solutions in a 4d gravitational model with a scalar field and two Abelian vector fields is presented. The scalar field is either ordinary ( $\varepsilon = +1$ ) or phantom ( $\varepsilon = -1$ ). The model contains two dilatonic coupling constants  $\lambda_s \neq 0$ ,  $s = 1, 2$ , obeying  $\lambda_1 \neq -\lambda_2$ .

The solutions are defined up to two moduli functions  $H_1(R)$  and  $H_2(R)$ , which obey two differential equations of second order with boundary conditions imposed. For  $\varepsilon = +1$  these equations are integrable for four cases, corresponding to the Lie algebras  $A_1 + A_1$ ,  $A_2$ ,  $B_2 = C_2$  and  $G_2$ . In the first case ( $A_1 + A_1$ ) we have  $\lambda_1\lambda_2 = 1/2$ , while in the second one ( $A_2$ ) we get  $\lambda_1 = \lambda_2 = \lambda$  and  $\lambda^2 = 3/2$ . Two other solutions, corresponding to the Lie algebras  $B_2 = C_2$  and  $G_2$ , will be considered in separate publications.

There is also a special solution with dependent electric and magnetic charges:  $\lambda_1 Q_1^2 = \lambda_2 Q_2^2$ , which is defined for all (admissible)  $\lambda_s$  and  $\varepsilon$  obeying  $\lambda_1\lambda_2 > 0$ .

Here we have also calculated some physical parameters of the solutions: gravitational mass  $M$ , scalar charge  $Q_\varphi$ , Hawking temperature, black hole area entropy and post-Newtonian

parameters  $\beta$ ,  $\gamma$ . The PPN parameters  $\gamma = 1$  and  $\beta$  do not depend upon  $\lambda_s$  and  $\varepsilon$ , if the values of  $M$  and  $Q_\varphi$  are fixed.

We have also obtained a formula, which relates  $M$ ,  $Q_\varphi$ , the dyon charges  $Q_1$ ,  $Q_2$ , and the extremality parameter  $\mu$  for all values of  $\lambda_s \neq 0$ . Remarkably, this formula does not contain  $\lambda_s$  and coincides with that of Ref. [1]. As in the case  $\lambda_1 = \lambda_2$ , the product of the Hawking temperature and the Bekenstein–Hawking entropy does not depend upon  $\varepsilon$ ,  $\lambda_s$  and the moduli functions  $H_s(R)$ .

Here we have obtained lower bounds on the gravitational mass and upper bounds on the scalar charge for  $1 + 2\lambda_s^2\varepsilon > 0$ , which are based on the conjecture (from Sect. 5) on the parameters of solutions  $P_1 = P_1(Q_1^2, Q_2^2)$ ,  $P_2 = P_2(Q_1^2, Q_2^2)$ . In [1] we have presented several results of numerical calculations which support our bounds for  $\lambda_1 = \lambda_2$ . A rigorous proof of this conjecture may be the subject of a separate publication. For  $\varepsilon = +1$  the lower bound on the gravitational mass is in agreement for  $\lambda_1 = \lambda_2$  with that obtained earlier by Gibbons et al. [11] by using certain spinor techniques.

It was noted in Sect. 3.3 that for  $\lambda_1 \neq \lambda_2$  there exist two integrable cases corresponding to the Lie algebras  $C_2$  and  $G_2$ , which will be analyzed in separate papers. They do not occur for  $\lambda_1 = \lambda_2$ .

An open question here is to find the conditions on the dilatonic coupling constants  $\lambda_s$  which guarantee the existence of the second (hidden) horizon and the existence of the extremal black hole in the limit  $\mu = +0$ . For  $\varepsilon = +1$ ,  $\lambda_1 = \lambda_2$  this problem was analyzed in Refs. [14, 28]. This question can be addressed to a separate publication.

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