# Comparing Nontriviality For E And EXP

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**Abstract**

A set $A$ is nontrivial for the linear-exponential-time class $\mathrm{E} = \mathrm{DTIME}(2^{lin})$ if for any $k \geq 1$ there is a set $B_k \in \mathrm{E}$ such that $B_k$ is ($p$-$m$-)reducible to $A$ and $B_k \notin \mathrm{DTIME}(2^{k \cdot n})$. I.e., intuitively, $A$ is nontrivial for E if there are arbitrarily complex sets in E which can be reduced to $A$. Similarly, a set $A$ is nontrivial for the polynomial-exponential-time class $\mathrm{EXP} = \mathrm{DTIME}(2^{poly})$ if for any $k \geq 1$ there is a set $\hat{B}_k \in \mathrm{EXP}$ such that $\hat{B}_k$ is reducible to $A$ and $\hat{B}_k \notin \mathrm{DTIME}(2^{n^k})$. We show that these notions are independent, namely, there are sets $A_1$ and $A_2$ in E such that $A_1$ is nontrivial for E but trivial for EXP and $A_2$ is nontrivial for EXP but trivial for E. In fact, the latter can be strengthened to show that there is a set $A \in \mathrm{E}$ which is weakly EXP-hard in the sense of Lutz [12] but E-trivial.

## 1 Introduction

The standard way for proving a problem to be intractable is to show that the problem is hard or complete for the linear-exponential-time class $\mathrm{E} = \mathrm{DTIME}(2^{lin})$ under polynomial-time-bounded many-one reducibility ($p$-$m$-reducibility for short). While the classical approach for extending this method is to allow more general polynomial-time reducibilities in the definition of hardness, Lutz [12] proposed an alternative generalization of this approach by relaxing hardness in a different direction. While a set $A$ is hard for a class C if *all* problems in C can be reduced to $A$ and complete if it is hard and a member of C, Lutz proposed to call a set $A$ weakly hard if a *nonnegligible* part of C can be reduced to $A$ and to call $A$ weakly complete if in addition $A \in \mathrm{C}$. For the exponential-time classes E and $\mathrm{EXP} = \mathrm{DTIME}(2^{poly})$, Lutz formalized these ideas by introducing resource-bounded measures on these classes and by saying that a subclass of E is negligible if it has measure 0 in E (and similarly for EXP).

A certain drawback of Lutz's weak hardness notion, called measure-hardness in the following, is that it is based on the somewhat technical concept of resource-bounded measure. So in [2] the authors suggested some alternative weak hardness notion, called nontriviality, which is conceptually much simpler and is solely based on the basic concepts of computational complexity theory. Here a subclass of E is considered to be negligible if it is contained in a finite level $\mathrm{E}_k = \mathrm{DTIME}(2^{kn})$ of the hierarchy E and, similarly, a subclass of EXP is considered to be negligible if it is contained in a finite level $\mathrm{EXP}_k = \mathrm{DTIME}(2^{n^k})$ of the hierarchy EXP. So nontrival sets for E and EXP have arbitrarily complex sets from E and EXP, respectively, among their predecessors.

As argued in [2], E-nontriviality may be viewed as the weakest weak hardness notion for E which is still reflecting the structure of E (and similarly for EXP). In particular, Lutz's measure hardness implies nontriviality. In [2] another weak hardness notion for E and EXP, called strong nontriviality, is introduced to fill the gap between measure-hardness and nontriviality, where strong nontriviality is obtained from nontriviality by replacing infinitely-often complexity by almost-everywhere complexity.

While, by a simple padding argument, hardness for E and EXP coincide, Juedes and Lutz [10] have shown that measure hardness for E implies measure hardness for EXP but that the converse is not true. Here we analyze the relations between the nontriviality notions for E and EXP. We show that nontriviality for E and nontriviality for EXP are independent: There are sets $A_1$ and $A_2$ in E such that $A_1$ is nontrivial for E but trivial for EXP and $A_2$ is nontrivial for EXP but trivial for E. In fact, we improve the latter by showing that there is an EXP-measure hard set $A_2 \in \mathrm{E}$ which is E-trivial. Moreover, we show that for strong nontriviality - just as in case of measure hardness - strong nontriviality for E implies strong nontriviality for EXP, thereby completely specifying the relations among all of the weak hardness notions considered here for E and EXP.

Note that the results on E- and EXP-nontriviality give some limitations on the padding technique. Recall that, by the Padding Lemma, for any set $A \in \mathrm{EXP}$ there is a set $\hat{A} \in \mathrm{E}_1$ which is $p$-$m$-equivalent (hence $p$-$m$-reducible) to $A$. (Namely, for $A \in \mathrm{EXP}_k$, it suffices to let $\hat{A} = \{0^{|x|^k}x : x \in A\}$.) So any set in EXP is $p$-$m$-equivalent to a set in the lowest level of the linear-exponential-time hierarchy. By a straightforward modification of this argument, one can show that any set $A \in \mathrm{E}_{k+1} \setminus \mathrm{E}_k$ from the $(k+1)$th level of E is $p$-$m$-equivalent to sets at all lower levels of E (namely, for any $j \leq k$ there is a set $A_j \in \mathrm{E}_{j+1} \setminus \mathrm{E}_j$ which is $p$-$m$-equivalent to $A$) (see [2]). This easily implies that a set $A$ is E-nontrivial if and only if $A$ is $p$-$m$-equivalent to sets at all levels $\mathrm{E}_{j+1} \setminus \mathrm{E}_j$ of E and, similarly, $A$ is EXP-nontrivial if and only if $A$ is $p$-$m$-equivalent to sets at all levels $\mathrm{EXP}_{j+1} \setminus \mathrm{EXP}_j$ of EXP. On the other hand, by the existence of E-trivial sets $A$ at all levels $\mathrm{E}_{k+1} \setminus \mathrm{E}_k$ of E (see [2]), there are sets in arbitrarily high levels of E which are not $p$-$m$-equivalent to any sets in any higher levels (and, similarly, for EXP). Now our new results on the independence of nontriviality for E and nontriviality for EXP show that, for a set $A \in \mathrm{E}$, being equivalent to sets at all levels of the E-hierarchy (EXP-hierarchy) will not imply that $A$ has predecessors at all levels of the EXP-hierarchy (E-hierarchy).

Our notation is standard (see e.g. the monographs of Balcázar et al. [6] and [7]). Moreover, we assume familiarity with the basic notions and results of computational complexity theory.

## 2 Weak Hardness Notions For E and EXP

In this section we introduce the weak hardness notions for E and EXP we will deal with. We start with some notation and simple facts on the exponential time classes. Let

$$\mathrm{E}_k = \mathrm{DTIME}(2^{kn}) \text{ and } \mathrm{EXP}_k = \mathrm{DTIME}(2^{k^n}).$$

Then

$$E = \bigcup_{k \geq 1} \mathrm{E}_k \text{ and } \mathrm{EXP} = \bigcup_{k \geq 1} \mathrm{EXP}_k.$$

Obviously, $\mathrm{E}_1 = \mathrm{EXP}_1$ and $\mathrm{E} \subseteq \mathrm{EXP}_2$. Moreover, by the Time-Hierarchy Theorem, $\mathrm{E}_k \subset \mathrm{E}_{k+1}$ and $\mathrm{EXP}_k \subset \mathrm{EXP}_{k+1}$ whence, in particular, for any $k \geq 1$, $\mathrm{E} \setminus \mathrm{E}_k$ and $\mathrm{EXP} \setminus \mathrm{EXP}_k$ are nonempty. So E and EXP may be viewed as hierarchies, the linear-exponential-time hierarchy and the polynomial-exponential-time hierarchy, respectively. By the Padding Lemma, for any set $A \in \mathrm{EXP}$ there is a set $\hat{A}$ in $\mathrm{E}_1$ such that $\hat{A}$ is $p$-$m$-equivalent to $A$. So hardness for E and EXP coincide, and a set $A \in \mathrm{E}$ is E-complete if and only if it is EXP-complete. Moreover, since EXP is downward closed under $p$-$m$-reducibility, it follows that

$$\mathrm{P}_m(\mathrm{E}_1) = \mathrm{P}_m(\mathrm{E}) = \mathrm{P}_m(\mathrm{EXP}) = \mathrm{EXP}$$

(where $\mathrm{P}_m(\mathrm{C}) = \{B : \exists\, A \in \mathrm{C}\ (B \leq^p_m A)\}$ denotes the downward closure of the class C under $p$-$m$-reducibility).

**Definition 2.1 (Ambos-Spies and Bakibayev [2])** *A set $A$ is* E-nontrivial *if, for any number $k \geq 1$, there is a set $B_k \in \mathrm{E} \setminus \mathrm{E}_k$ such that $B_k \leq^p_m A$; and $A$ is* E-trivial *otherwise.*

*A set $A$ is* EXP-nontrivial *if, for any number $k \geq 1$, there is a set $B_k \in \mathrm{EXP} \setminus \mathrm{EXP}_k$ such that $B_k \leq^p_m A$; and $A$ is* EXP-trivial *otherwise.*

So, intuitively, a set $A$ is E-nontrivial if there are arbitrarily complex linear-exponential-time sets which can be reduced to $A$, i.e., if for each $k$ there is a linear-exponential-time set $B_k$ $p$-$m$-reducible to $A$ such that the run time of any algorithm computing $B_k$ exceeds $2^{k|x|}$ on infinitely many strings $x$ (and similarly for EXP). In the following definition nontriviality is strengthened by replacing infinitely-often complexity by almost-everywhere complexity. Here we use the coincidence of almost-everywhere complexity and bi-immunity (see e.g. Chapter 6 of Balcázar et al. [7]).

**Definition 2.2 (Ambos-Spies and Bakibayev [2])** *A set $A$ is* strongly E-nontrivial *if, for any number $k \geq 1$, there is an $\mathrm{E}_k$-bi-immune set $B_k \in \mathrm{E}$ such that $B_k \leq^p_m A$; and $A$ is* weakly E-trivial *otherwise.*

*A set $A$ is* strongly EXP-nontrivial *if, for any number $k \geq 1$, there is an $\mathrm{EXP}_k$-bi-immune set $B_k \in \mathrm{EXP}$ such that $B_k \leq^p_m A$; and $A$ is* weakly EXP-trivial *otherwise.*

The third type of weak hardness we consider here, the original weak hardness notion of Lutz [12], is technically more involved by being based on the resource-bounded measure theory developed in Lutz [11]. So, in order to explain Lutz's weak hardness notion, which we call *measure hardness* here, we have to introduce the basic concepts of resource-bounded measure theory first. For simplicity, we take the characterization of measure hardness in terms of time-bounded randomness given in [5] as definition here. Though this characterization is less intuitive, it is more simple and more convenient for applications.

For the motivation underlying measure hardness we refer to Lutz [12]. More background information on resource-bounded measure theory can be found in the surveys on resource-bounded measure by Lutz [13] and Ambos-Spies and Mayordomo [4] where the latter also explains the relations between resource-bounded measure and randomness. The definitions and facts needed here are taken from [4].

In the following definition we summarize the concepts from resource-bounded measure theory we will need.

**Definition 2.3** *(a) A* martingale *is a real valued function* $d : \{0,1\}^* \to [0,\infty)$ *such that* $d(\lambda) > 0$ *and, for every* $x \in \{0,1\}^*$*, the following equality (called* fairness condition*) holds.*

$$\frac{d(x0) + d(x1)}{2} = d(x) \tag{1}$$

$d(\lambda)$ *is called the* norm *of* $d$*.* $d$ *is* normed *if* $d(\lambda) = 1$*.*

*(b) A martingale* $d$ succeeds *on a set* $A$ *if*

$$\limsup_{n \geq 0} d(A \upharpoonright n) = \infty$$

*(where* $A \upharpoonright n = A(0), \ldots, A(n-1)$ *is the initial segment of length* $n$ *of the characteristic sequence of* $A$*). A martingale* $d$ succeeds *on a class* C *if it succeeds on all sets* $A \in$ C*.*

*(c) The* (betting) strategy $s_d$ underlying the martingale $d$ *is the function*

$$s_d(x) = \begin{cases} \frac{d(x0)}{2d(x)} & \text{if } d(x) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

*(d) A* $t(n)$-martingale $d$ *is a rational valued martingale* $d : \{0,1\}^* \to \mathbb{Q} \cap [0,\infty)$ *such that, for the underlying strategy* $s_d$*,* $s_d \in \mathrm{DTIME}(t(n))$*.*

*(e) A class* C *has* $t(n)$-measure *0 if there is a* $t(n)$*-martingale which succeeds on* C*.*

*(f) A set* $A$ *is* $t(n)$-random *if no* $t(n)$*-martingale succeeds on* $A$ *(i.e., if the singleton* $\{A\}$ *does not have* $t(n)$*-measure 0).*

In addition to these concepts, below we will use the following observations on the complexity of time-bounded martingales and on universal martingales which are special cases of more general results on time-bounded martingales in [4].

**Lemma 2.4** *For any* $n^k$*-martingale* $d$*,* $d \in \mathrm{DTIME}(n^{k+2})$ *(*$k \geq 1$*).*

**Lemma 2.5** *For* $k \geq 1$ *there is an* $n^{k+4}$*-martingale* $d$ *which succeeds on all* $n^k$*-measure-0 classes (i.e., which succeeds on all sets on which any* $n^k$*-martingale succeeds).*

Having introduced the basic concepts of resource-bounded randomness, we can now define Lutz's weak hardness notions for E and EXP.

**Definition 2.6 (Lutz [12])** *A set $A$ is* E-measure hard *if, for any number $k \geq 1$, there is an $n^k$-random set $B_k \in \mathrm{E}$ such that $B_k \leq_m^p A$. And $A$ is* E-measure complete *if $A \in \mathrm{E}$ and $A$ is* E*-measure hard.*

*A set $A$ is* EXP-measure hard *if, for any number $k \geq 1$, there is a $2^{(\log n)^k}$-random set $B_k \in \mathrm{EXP}$ such that $B_k \leq_m^p A$. And $A$ is* EXP-measure complete *if $A \in \mathrm{EXP}$ and $A$ is* EXP*-measure hard.*

By some simple observation on random sets, Ambos-Spies, Terwijn and Zheng [5] have given a very useful and simple characterization of measure hardness for E and EXP. Moreover, by some similar observation on bi-immune sets, Ambos-Spies and Bakibayev [2] have given a corresponding characterization of strong nontriviality for E and EXP.

**Lemma 2.7 (Ambos-Spies, Terwijn and Zheng [5])** *Let $A$ be an $n^2$-random set, and, for $k \geq 1$, let*

$$A_k = \{x : 0^{k \cdot |x|} x \in A\} \quad \text{and} \quad A'_k = \{x :\ 0^{|x|^{k+1}} x \in A\}. \tag{2}$$

*Then $A_k \leq_m^p A$, $A'_k \leq_m^p A$, $A_k$ is $n^k$-random, and $A'_k$ is $2^{(log\, n)^k}$-random. Moreover, if $A \in \mathrm{E}$ then $A_k \in \mathrm{E}$ too.*

**Theorem 2.8 (Characterization Theorem for Measure Hardness, [5])** *A set $A$ is* E*-measure hard if and only if there is an $n^2$-random set $B \in \mathrm{E}$ such that $B \leq_m^p A$. And a set $A$ is* EXP*-measure hard if and only if there is an $n^2$-random set $B \in \mathrm{EXP}$ such that $B \leq_m^p A$.*

**Lemma 2.9 (Ambos-Spies and Bakibayev [2])** *Let $A$ be an $\mathrm{E}_1$-bi-immune set. Then, for $k \geq 1$ and for $A_k$ and $A'_k$ as in (2), $A_k \leq_m^p A$, $A'_k \leq_m^p A$, $A_k$ is $\mathrm{E}_k$-bi-immune, and $A'_k$ is $\mathrm{EXP}_k$-bi-immune. Moreover, if $A \in \mathrm{E}$ then $A_k \in \mathrm{E}$ too.*

**Theorem 2.10 (Characterization Theorem for Strong Nontriviality, [2])** *A set $A$ is strongly* E*-nontrivial if and only if there is an $\mathrm{E}_1$-bi-immune set $B \in \mathrm{E}$ such that $B \leq_m^p A$. And a set $A$ is strongly* EXP*-nontrivial if and only if there is an $\mathrm{E}_1$-bi-immune set $B \in \mathrm{EXP}$ such that $B \leq_m^p A$.*

# 3 Some Relations Among the Weak Hardness Notions

We now summarize the relations among the weak hardness notions for E and EXP which can be found in the literature.

Since, for any $k \geq 1$, there is an $n^k$-random set in E, since any $n^{k+1}$-random set is $\mathrm{E}_k$-bi-immune and since no $\mathrm{E}_k$-bi-immune set is in $\mathrm{E}_k$, we obtain the following relations among the weak hardness notions for E (and, similarly, for EXP).

**Lemma 3.1 ([12], [2])** *For any set $A$ and for $\mathrm{C} \in \{\mathrm{E}, \mathrm{EXP}\}$,*

$$\begin{array}{c} A \text{ C-}\textit{hard} \\ \Downarrow \\ A \text{ C-}\textit{measure hard} \\ \Downarrow \\ A \textit{ strongly } \text{C-}\textit{nontrivial} \\ \Downarrow \\ A \text{ C-}\textit{nontrivial} \end{array} \tag{3}$$

In fact, the implications in (3) are strict as the following shows.

**Theorem 3.2** *(a) There is an* $\mathrm{E}_1$*-bi-immune (hence* P*-bi-immune) set* $A \in \mathrm{E}$ *which is* E*-measure hard and* EXP*-measure hard (Ambos-Spies, Terwijn, Zheng [5]) whereas no* E*-hard set is* P*-bi-immune (Berman [8]).*

*(b) There is a tally set* $A$ *in* E *which is strongly* E*-nontrivial and strongly* EXP*-nontrivial (Ambos-Spies and Bakibayev [2]) whereas any* E*-measure hard or* EXP*-measure hard set is exponentially dense (Lutz and Mayordomo [14]).*

*(c) There is an exptally set* $A$ *in* E *which is* E*-nontrivial and* EXP*-nontrivial whereas no strongly* E*-nontrivial or strongly* EXP*-nontrivial set is exptally (Ambos-Spies and Bakibayev [2]).*

Here a set $A$ is tally if it is a subset of $\{0\}^*$, and $A$ is exptally if $A \subseteq \{0^{\delta(n)} : n \geq 0\}$ where $\delta : \mathbb{N} \to \mathbb{N}$ is the iterated exponential function inductively defined by $\delta(0) = 0$ and $\delta(n+1) = 2^{\delta(n)}$.

Having established the hierarchy of weak hardness notions, we now turn to the relations between weak hardness for E and weak hardness for EXP for the individual weak hardness notions. While, as pointed out above, E-hardness and EXP-hardness coincide by the Padding Lemma, Juedes and Lutz have shown that measure hardness for E implies measure hardness for EXP but that the converse in general fails.

**Theorem 3.3 (Juedes and Lutz [10])** *Every* E*-measure hard set is* EXP*-measure hard. But there is an* EXP*-measure hard set* $A \in \mathrm{E}$ *which is not* E*-measure hard.*

Note that the positive implication in Theorem 3.3 is immediate by the Characterization Theorem for Measure Hardness (Theorem 2.8). Similarly, the Characterization Theorem for Strong Nontriviality (Theorem 2.10) immediately implies that any strongly E-nontrivial set is strongly EXP-nontrivial.

**Lemma 3.4 (Ambos-Spies and Bakibayev [2])** *Every strongly* E*-nontrivial set is strongly* EXP*-nontrivial.*

In the following two sections we answer the questions about the relations between weak hardness for E and EXP left open by the above results.

# 4 An EXP-Trivial E-Nontrivial Set

In contrast to the weak hardness notions of measure hardness and strong nontriviality where weak hardness for E implies weak hardness for EXP, there are E-nontrivial sets which are not EXP-nontrivial.

**Theorem 4.1** *There is a set* $A \in \mathrm{E}$ *such that* $A$ *is* E*-nontrivial and* EXP*-trivial.*

Proof. We construct a set $A \in \mathrm{E}$ with the required properties. In order to make $A$ E-nontrivial and EXP-trivial we satisfy the conditions

$$\forall\, k \geq 1\ (\{0^{k|x|}x : x \in \Sigma^*\} \cap A \notin \mathrm{E}_1) \tag{4}$$

and

$$\forall\, f \in \mathrm{P}\ \forall^\infty x\ (|f(x)| \geq |x|^2 \Rightarrow f(x) \notin A), \tag{5}$$

respectively.

Note that, for $A_k = \{0^{k|x|}x : x \in A\}$, (4) implies that $A_k \notin \mathrm{E}_k$. Since (as one can easily check or by Lemma 2.7) $A_k \leq^p_m A$ and, assuming $A \in \mathrm{E}$, $A_k \in \mathrm{E}$, it follows that any set $A \in \mathrm{E}$ satisfying (4) is E-nontrivial. In order to show that, for $A \in \mathrm{E}$, (5) implies that $A$ is EXP-trivial, it suffices to show that, for any set $B \leq^p_m A$, $B \in \mathrm{EXP}_4$. Fix $f$ such that $B \leq^p_m A$ via $f$. Then, given a string $x$, $B(x)$ can be computed in $O(2^{|x|^4})$ steps by using the identity $B(x) = A(f(x))$. Namely, if $|f(x)| \geq |x|^2$ then, by (5), $B(x) = A(f(x)) = 0$ while in case of $|f(x)| < |x|^2$, it follows from $A \in \mathrm{E} \subseteq \mathrm{EXP}_2$ that $A(f(x))$ can be computed in $O(2^{|f(x)|^2}) \leq O(2^{(|x|^2)^2}) = O(2^{|x|^4})$ steps.

For satisfying (4) and (5) we break up these conditions into infinitely many requirements. Fix computable enumerations $\{E^1_n : n \geq 0\}$ and $\{f_n : n \geq 0\}$ of $\mathrm{E}_1$ and the class of the polynomial-time computable functions, respectively, such that, for $x$ with $|x| \geq n$, $E^1_n(x)$ and $f_n(x)$ can be uniformly computed in $2^{3|x|}$ and $2^{|x|}$ steps, respectively. Moreover, let $\langle k, m\rangle$ be a standard polynomial-time computable pairing function satisfying $k, m \leq \langle k, m\rangle$. Then, in order to satisfy (4) and (5), it suffices to meet the requirements

$$\mathcal{P}_{\langle k,m\rangle} : \exists x\ (A(0^{k|x|}x) \neq E^1_m(0^{k|x|}x))$$

$$\mathcal{N}_e : \forall x^\infty\ (|f_e(x)| \geq |x|^2 \Rightarrow f_e(x) \notin A)$$

for all numbers $k \geq 1$ and $m, e \geq 0$.

Now the basic strategies for meeting these requirements are as follows. The $\mathcal{P}$-requirements are met by diagonalization. We will pick strings $x_0 < x_1 < x_2 < \dots$ such that, for the corresponding strings $y_n$ defined by

$$y_{\langle k,m\rangle} = 0^{k|x_{\langle k,m\rangle}|}x_{\langle k,m\rangle},$$

$y_0 < y_1 < y_2 < \dots$, and we will let

$$A = \{y_{\langle k,m\rangle} : k \geq 1\ \&\ m \geq 0\ \&\ y_{\langle k,m\rangle} \notin E^1_m\}. \tag{6}$$

So string $x_{\langle k,m\rangle}$ will witness that requirement $\mathcal{P}_{\langle k,m\rangle}$ is met.

In order to meet requirement $\mathcal{N}_e$ it suffices to ensure that at most finitely many *e-forbidden* strings $y$ are put into $A$ where a string $y$ is $e$-forbidden if

$$\exists x < y\ (|x|^2 < |y|\ \&\ f_e(x) = y).$$

This is achieved by ensuring that, for $n > e$, the string $x_n$ is chosen so that the corresponding string $y_n$ is not $e$-forbidden whence, by (6), the only $e$-forbidden strings which may enter $A$ are the strings $y_0, \dots, y_e$.

So, in order to complete the proof, it only remains to choose the strings $x_n$ in such a way that the corresponding strings $y_n$ are not $e$-forbidden for $e < n$ and such that, for the corresponding set $A$ defined by (6), $A \in \mathrm{E}$.

We first inductively define the length $l_n$ of string $y_n$. Let $l_0 = 0$ (i.e., $y_0 = x_0 = \lambda$). For $n > 0$ fix $k, m$ such that $n = \langle k, m \rangle$, and let $l_n$ be the least number $l > l_{n-1}$ such that $l = (k+1)^2 \cdot r^2$ for some number $r > n$.

Then, given $n > 0$, $x_n$ is defined as follows. Fix $k, m \leq n$ and $r > n$ such that $n = \langle k, m \rangle$ and $l_n = (k+1)^2 \cdot r^2$. Let $F_n$ be the set of all strings $y$ of length $l_n$ which are $e$-forbidden for some $e < n$. Note that $|F_n| < n \cdot 2^{(k+1)r}$ since there are $n$ numbers $e < n$ and, for any $e$ and for any $e$-forbidden string $y$ of length $l_n = ((k+1)r)^2$, the preimage of $y$ under $f_e$ has length less than $(k+1)r$ whence there are less than $2^{(k+1)r}$ $e$-forbidden strings of length $l_n$. Since $1 \leq n < r$ it follows that

$$|\{0,1\}^{(k+1)r^2}| = 2^{(k+1)r^2} > n \cdot 2^{(k+1)r} > |F_n|.$$

So there is a string $x$ of length $(k+1)r^2$ such that $0^{k|x|}x \notin F_n$ and we may let $x_n$ be the least such string $x$. Obviously, $x_n$ has the required properties.

Finally, in order to show that $A \in \mathrm{E}$, first observe that, for a given string $y$, in $O(2^{4|y|})$ steps we can decide whether $y = y_n$ for some $n \geq 1$ and if so determine the corresponding $n$. Namely, in $poly(|y|)$ steps we can decide whether $|y| = l_n$ for some $n = \langle k, m \rangle \geq 1$ and, if so, decide whether $y = 0^{k|x|}x$ for some $x$. Moreover, for a string $y'$ of length $l_n$ and for $e < n$ we can decide in $O(2^{2l_n})$ steps whether $y'$ is $e$-forbidden, hence in $O(n \cdot 2^{2l_n}) \leq O(2^{3l_n})$ steps whether $y' \in F_n$. Since $y_n$ is the least string $y$ of length $l_n$ such that $y = 0^{k|x|}x$ for some $x$ and such that $y \notin F_n$, it follows that $y = y_n$ can be decided in $O(2^{4|y|})$ steps. Since, for given $y = y_n$ where $n = \langle k, m \rangle \leq |y|$, $E^1_m(y)$ can be computed in time $2^{3|y|}$, it follows, by (6), that $A \in \mathrm{E}_4$.

This completes the proof. □

# 5 An E-Trivial EXP-Measure Complete Set

We now come to our main result. We show that there is an EXP-measure hard set in E which is not E-nontrivial. Since EXP-measure hardness is the strongest weak hardness notion for EXP while E-nontriviality is the weakest weak hardness notion for E, this implies that none of the weak hardness notions for EXP implies any of the weak hardness notions for E.

**Theorem 5.1** *There is an* EXP*-measure hard set in* E *which is* E*-trivial.*

Theorem 5.1 is an easy consequence of the following lemma on the existence of E-trivial $n^2$-random sets in EXP.

**Lemma 5.2** *There is an $n^2$-random set $A \in \mathrm{EXP}$ such that $A$ is* E*-trivial.*

Lemma 5.2 implies Theorem 5.1 as follows. By Lemma 5.2 fix $A \in \mathrm{EXP}$ such that $A$ is $n^2$-random and E-trivial. By the former and by Theorem 2.8, $A$ is EXP-measure hard. Now, by the Padding Lemma, let $\hat{A}$ be any set in $\mathrm{E}_1$ such that $\hat{A}$ is $p$-$m$-equivalent to $A$. Since, by definition, EXP-measure hardness and E-nontriviality are invariant under $p$-$m$-equivalence, the set $\hat{A} \in \mathrm{E}$ is EXP-measure hard and E-trivial.

Proof (of Lemma 5.2). By a slow diagonalization we inductively construct a set $A$ with the desired properties. At stage $s$ of the construction we determine the value

$A(z_s)$ of $A$ on the $s$th string $z_s$ (w.r.t. the length lexicographical ordering) and, at the same time, we satisfy the highest priority requirement $\Re_e$ (to be defined below) which has not yet been satisfied before and which can be satisfied by appropriately choosing the value of $A(z_s)$.

Before we give the formal construction, we first explain the basic strategies for achieving the two main goals of the construction, namely to make $A$ $n^2$-random and E-trivial, point out the conflicts between these strategies, and explain how these conflicts are resolved.

In order to make $A$ $n^2$-random, we have to ensure that no $n^2$-martingale succeeds on $A$. Since, by Lemma 2.5, there is a normed $n^6$-martingale $d$ which succeeds on all $n^2$-measure-0 classes, it suffices to fix such an $n^2$-universal martingale $d$ and to guarantee that $d$ does not succeed on $A$. We will achieve this by ensuring

$$\forall s\ (d(A \upharpoonright z_s) \leq 1). \tag{7}$$

In order to meet our second main goal, namely to make $A$ E-trivial, it suffices to ensure

$$\forall B \in \mathrm{E}\ (B \leq^p_m A \Rightarrow B \in DTIME(2^n)). \tag{8}$$

In order to satisfy (8), we will guarantee that any $p$-$m$-reduction $f$ of a set $B \in \mathrm{E}$ to $A$ sufficiently compresses $B$ so that, by using the identity $B(x) = A(f(x))$, we can compute $B(x)$ in $O(2^{|x|})$ steps. To achieve this, we have to destroy the $p$-$m$-reductions from sets in E to $A$ which are not sufficiently compressing by diagonalization. In order to ensure that the time required for these diagonalizations is compatible with making $A$ exponential-time computable, we will use a somewhat tricky strategy which is reminiscent of the diagonalization technique in the proof of Blum's speed-up theorem.

Let $\{f_e : e \geq 0\}$ be a computable enumeration of the class of the polynomial time computable functions such that, for uniformly given polynomial time bounds $p_e$ for $f_e$ ($e \geq 0$), $p_e(n^2) \leq 2^n$ for all $n > e$, and let $\{E_e : e \geq 0\}$ be a computable enumeration of the class E such that, for $x$ with $|x| > e$, $E_e(x)$ can be uniformly computed in time $2^{e \cdot |x|}$. Then (8) is split into the finitary requirements

$$\Re_e : E_{e_0} \leq^p_m A \ via\ f_{e_1} \Rightarrow \forall^\infty x(|x| > 2^{-e} \cdot |f_{e_1}(x)|^2)$$

where $e \geq 0$ and $e = \langle e_0, e_1 \rangle$. In addition, we ensure

$$\forall \alpha > 0\ (A \in \mathrm{DTIME}(2^{\alpha \cdot n^2})) \tag{9}$$

where $\alpha$ is a real number. (Of course this will a fortiori ensure that $A$ is in $\mathrm{EXP}_2$ hence in EXP.)

To show that the above will guarantee that $A$ satisfies (8), fix a set $B \in \mathrm{E}$ such that $B \leq^p_m A$. It suffices to show that $B \in \mathrm{DTIME}(2^n)$. Fix $e_0$ and $e_1$ such that $B = E_{e_0}$ and $B \leq^p_m A$ via $f_{e_1}$, and let $e = \langle e_0, e_1 \rangle$. Then, by requirement $\Re_e$, we may fix $n_0$ such that, for $\alpha = 2^{-e}$,

$$\forall\ x\ (|x| \geq n_0 \Rightarrow |x| > \alpha \cdot |f_{e_1}(x)|^2). \tag{10}$$

Now, given a string $x$ with $|x| \geq \max(e, n_0)$, $B(x)$ can be computed in time $O(2^n)$ (for $n = |x|$) as follows. Since $B(x) = A(f_{e_1}(x))$, it suffices to compute $y = f_{e_1}(x)$

and $A(y)$. The former can be done in $poly(n)$, hence in $O(2^n)$, steps. The latter can be done in $2^n$ steps as follows. By (10), $\alpha \cdot |y|^2 < n$. So, by (9), $A(y)$ can be computed in $2^{\alpha \cdot |y|^2} \leq 2^n$ steps.

Having isolated the properties of $A$ to be guaranteed by the construction, namely to satisfy condition (7) and to meet the requirements $\Re_e$ ($e \geq 0$) and at the same time ensure the time bounds given in (9), we next look at the strategies for satisfying (7) and the requirements $\Re_e$, respectively, and show how these strategies can be made to be compatible with each other.

The basic strategy for meeting the martingale equation (7) is quite simple. Note that, by the fairness property of martingales,

$$\frac{d((A \upharpoonright z_s)0) + d((A \upharpoonright z_s)1)}{2} = d(A \upharpoonright z_s). \tag{11}$$

So, for any $s \geq 0$, there is an $i \leq 1$ such that

$$d((A \upharpoonright z_s)i) \leq d(A \upharpoonright z_s). \tag{12}$$

Since $d$ is normed, i.e., $d(\lambda) = 1$, it follows that (7) can be trivially satisfied by letting

$$A(z_s) = i \text{ for some (say the least) } i \text{ such that } d((A \upharpoonright z_s)i) \leq d(A \upharpoonright z_s). \tag{13}$$

In the following we say that $A(z_s)$ is defined according to the *basic randomness strategy* if (13) holds.

The basic strategy for meeting a requirement $\Re_e$ ($e = \langle e_0, e_1 \rangle$) (in the following called the *basic $\Re_e$-strategy*) is as follows. Wait for a stage $s$ such that there is a string $x$ with $|x| \leq 2^{-e}|z_s|^2$ and $f_{e_1}(x) = z_s$. Then meet the requirement by letting

$$A(z_s) = 1 - E_{e_0}(x) \tag{14}$$

thereby ensuring that the hypothesis $E_{e_0} \leq^p_m A$ via $f_{e_1}$ of $\Re_e$ fails.

Of course it may happen that defining $A(z_s)$ according to (14) is not compatible with the basic randomness strategy since

$$d((A \upharpoonright z_s)(1 - E_{e_0}(x))) > d(A \upharpoonright z_s).$$

In order to resolve this conflict, we use some idea of Ambos-Spies and Kräling [3]. First note that we can relax the basic randomness strategy as follows. If $\Re_e$ wants to act at stage $s$ and wants to define $A(z_s)$ according to (14), this does not do any harm to ensuring (7) as long as

$$d((A \upharpoonright z_s)(1 - E_{e_0}(x))) \leq 1. \tag{15}$$

So, since there will be infinitely many stages $s$ such that $\Re_e$ can be met at stage $s$ by letting the basic $\Re_e$-strategy act as described above (unless $\Re_e$ is trivially met and no action becomes necessary), it suffices to ensure that, for one of these stages, (15) will hold. But this can be achieved by the following observation. Whenever we cannot meet $\Re_e$ at a stage $s$ since (15) fails then, by letting $d(A \upharpoonright z_{s+1}) = d(A \upharpoonright z_s)E_{e_0}(x)$, the value of $d$ is strictly decreased (by the fairness property of martingales). So, assuming that no other requirement is interfering with the definition of $d$, eventually the value of $d(A \upharpoonright z_s)$ will be so small that (15) will hold. In order to make sure that the decreases in values of the martingale $d$ on the initial segments of $A$ occurring at stages at which requirement $\Re_e$ is blocked from acting are not compensated by increases of $d$ caused by actions of some other requirements thereby blocking $\Re_e$ forever, we endow requirement $\Re_e$ with an account in which the amounts are accumulated by which $d$ is dropping at stages at which requirement $\Re_e$ is eligible to act but blocked. Then it is safe to relax the basic randomness strategy by letting $\Re_e$ act at stage $s$ according to (14) as long as

$$d((A \upharpoonright z_s)(1 - E_{e_0}(x))) \leq d(A \upharpoonright z_s) + b_e(s-1) \tag{16}$$

where $b_e(s-1)$ is the balance of the account of $\Re_e$ at the end of stage $s-1$. Moreover, as one can easily check, whenever $\Re_e$ is blocked (after the first time) the balance of $\Re_e$'s account is doubled. So, eventually, the balance of the account of $\Re_e$ will be high enough to allow $\Re_e$ to pay the prize for its action.

We conclude our discussion of the basic strategies underlying the construction of $A$ by explaining the reason why our strategy for meeting the requirements $\Re_e$ is compatible with satisfying (9). Here it is crucial to note that the bound for the search of a diagonalization witness is decreasing in $e$. So, since the requirements are finitary, we may speed-up the algorithm for computing $A$ given by the actual construction as follows. Use a finite table summarizing the impact of the first $e$ requirements on the construction and ignore these requirements in the construction otherwise. As we will show in the verification part of the proof following the formal construction, this sped-up versions of the construction will witness (9).

We now turn to the formal construction. Simultaneously with $A$ we define the balances $b_e(s)$ of the accounts of the requirements $\Re_e$. Moreover, we will determine which requirements $\Re_e$ require attention (if any) and which of these requirements will be eligible to act and, possibly, become active or satisfied.

<u>Stage 0.</u> Let $A(z_0)$ be the least $i \leq 1$ such that $d(i) \leq 1$. For $e \geq 0$, let $b_e(0) = 0$. Moreover, no requirement $\Re_e$ requires attention at stage 0, no requirement is eligible to act, and no requirement becomes active or satisfied.

<u>Stage $s > 0$.</u> Let $r_s$ be the least $i \leq 1$ such that (12) holds, and say that requirement $\Re_e$ *requires attention* at stage $s$ if $e < |z_s|$, $\Re_e$ has not been satisfied at any previous stage, and the following holds:

$$\exists x(|x| \leq 2^{-e} \cdot |z_s|^2 \ \& \ f_{e_1}(x) = z_s). \tag{17}$$

Now, if no requirement requires attention then let $A(z_s) = r_s$ and $b_e(s) = b_e(s-1)$ for all $e \geq 0$. Otherwise, fix $e$ minimal such that $\Re_e$ requires attention, declare that $\Re_e$ is *eligible to act* at stage $s$, fix $e_0, e_1$ such that $e = \langle e_0, e_1 \rangle$, fix the least number $x$ as in (17), let

$$i = 1 - E_{e_0}(x), \tag{18}$$

let $b_{e'}(s) = b_{e'}(s-1)$ for $e' \neq e$, and distinguish the following cases.

If

$$d((A \upharpoonright z_s)i) > d(A \upharpoonright z_s) + b_e(s-1) \tag{19}$$

then say that $\Re_e$ is *blocked* at stage $s$, and let $A(z_s) = r_s$ and

$$b_e(s) = b_e(s-1) + (d(A \upharpoonright z_s) - d(A \upharpoonright z_{s+1})). \tag{20}$$

Otherwise, let $A(z_s) = i$ and $b_e(s) = 0$, and say that $\Re_e$ is *active* and *satisfied* at stage $s$.

This completes the construction.

In order to show that the thus defined set $A$ has the required properties, we prove a series of claims.

*Claim 1. For $s \geq 0$, $b_e(s) \geq 0$ (for all $e \geq 0$) and*

$$d(A \restriction z_{s+1}) + \sum_{e \geq 0} b_e(s) \leq 1. \tag{21}$$

*Proof.* The proof is by induction on $s$. For $s = 0$, $b_e(0) = 0$ for all $e \geq 0$ and, by choice of $A(z_0)$, $d(A \restriction z_{s+1}) = d(A(z_0)) \leq d(\lambda) = 1$. For $s > 0$, distinguish the following cases.

If no requirement requires attention then $b_e(s) = b_e(s-1)$ for all $e \geq 0$ and $A(z_s)$ is chosen so that $d(A \restriction z_{s+1}) \leq d(A \restriction z_s)$. So the claims are immediate by inductive hypothesis.

If requirement $\Re_e$ is eligible to act at stage $s$ but blocked, then, by construction, $d(A \restriction z_{s+1}) < d(A \restriction z_s)$ and (by (20))

$$d(A \restriction z_{s+1}) + b_e(s) = d(A \restriction z_s) + b_e(s-1)$$

while $b_{e'}(s) = b_{e'}(s-1)$ for $e' \neq e$. So, in particular, $b_e(s) > b_e(s-1)$ and

$$d(A \restriction z_{s+1}) + \sum_{e \geq 0} b_e(s) = d(A \restriction z_s) + \sum_{e \geq 0} b_e(s-1)$$

whence the claims follow by inductive hypothesis.

Finally, if requirement $\Re_e$ becomes active at stage $s$ then, by construction,

$$d(A \restriction z_{s+1}) \leq d(A \restriction z_s) + b_e(s-1)$$

and $b_e(s) = 0$ while $b_{e'}(s) = b_{e'}(s-1)$ for $e' \neq e$. So, again, the claims are immediate by inductive hypothesis.

*Claim 2. $A$ is $n^2$-random.*

*Proof.* By Claim 1, $A$ satisfies (7). So $A$ is $n^2$-random by choice of $d$.

*Claim 3. Every requirement $\Re_e$ requires attention at most finitely often.*

*Proof.* For a contradiction, pick $e$ minimal such that requirement $\Re_e$ requires attention infinitely many times. By minimality of $e$, we may fix $s^*$ such that no requirements $\Re_{e'}$ with $e' < e$ will require attention after stage $s^*$. Then, whenever $\Re_e$ requires attention after stage $s^*$, $\Re_e$ will be eligible to act. On the other hand, $\Re_e$ will never become active since once a requirement became active it stops to require attention.

So there are infinitely many stages at which $\Re_e$ is eligible to act and $\Re_e$ becomes blocked at all of these stages. Let $s_0 < s_1 < s_2 < \ldots$ be these stages. Now, by a straightforward induction on $s$, $0 \leq b_e(s) \leq b_e(s+1)$ (since $\Re_e$ is never active).

Moreover, for any stage $s_n$ $(n \geq 0)$, $A(z_{s_n}) = 1 - i$ for some $i \leq 1$ satisfying (19) whence, by the fairness condition (11),

$$d(A \upharpoonright z_{s_n+1}) < d(A \upharpoonright z_{s_n}) - b_e(s_n - 1).$$

So, by (20), $b_e(s_0) > 0$ and, for $n \geq 1$, $b_e(s_n) > 2 \cdot b_e(s_n - 1) \geq 2 \cdot b_e(s_{n-1})$. It follows that

$$\lim_{s \to \infty} b_e(s) = \lim_{n \to \infty} b_e(s_n) = \infty.$$

Since $d$ is a martingale, hence nonnegative, this contradicts (21) in Claim 1.

*Claim 4. Every requirement $\Re_e$ is met.*

*Proof.* For a contradiction assume that requirement $\Re_e$ is not met. Fix $e_0, e_1$ such that $e = \langle e_0, e_1 \rangle$. Then $E_{e_0} \leq_m^p A$ via $f_{e_1}$ and

$$\exists^\infty x \ (|x| < 2^{-e} \cdot |f_{e_1}(x)|^2). \tag{22}$$

Moreover, $\Re_e$ is never satisfied. (Obviously, if $\Re_e$ becomes satisfied at a stage $s$ then the hypothesis of $\Re_e$ fails whence $\Re_e$ is met.) So $\Re_e$ requires attention at any stage $s$ such that $e < |z_s|$ and (17) holds. But, as one can easily show, by (22) there will be infinitely many such stages $s$. So, contrary to Claim 3, $\Re_e$ requires attention infinitely often.

It remains to show that $A$ satisfies (9). In order to do so we have to analyze the complexity of the basic features of the construction. Recall that $d$ is an $n^6$-martingale whence, by Lemma 2.4, $d \in \mathrm{DTIME}(n^8)$. So, given $A \upharpoonright z_s$, $d(A \upharpoonright z_s)$ can be computed in $O(2^{8 \cdot |z_s|})$ steps.

*Claim 5. Given $e, e_0, e_1, s \geq 0$ such that $e = \langle e_0, e_1 \rangle$ and $6 < e < |z_s|$, the following can be done in $O(2^{\frac{1}{e+1}|z_s|^2})$ steps: decide whether (17) holds and, if so, compute the least witness $x$ for (17) and decide whether $x \in E_{e_0}$.*

*Proof.* It suffices to look at all strings $x$ with

$$|x| \leq 2^{-e} \cdot |z_s|^2, \tag{23}$$

and to compute $f_{e_1}(x)$ and $E_{e_0}(x)$ for each such $x$. Now, by (23) and by choice of $\{f_m : m \geq 0\}$, $f_{e_1}(x)$ can be computed in $O(2^{|z_s|})$ steps, while, by (23) and by choice of $\{E_m : m \geq 0\}$, $E_{e_0}(x)$ can be computed in

$$O(2^{e_0 |x|}) \leq O(2^{e_0 (2^{-e} \cdot |z_s|^2)})$$

steps.

Since there are $O(2^{2^{-e} \cdot |z_s|^2})$ strings $x$ as in (17), the above procedure can be completed in

$$O(2^{2^{-e} \cdot |z_s|^2}) \cdot (O(2^{|z_s|}) + O(2^{e_0 (2^{-e} \cdot |z_s|^2)})) \leq O(2^{\frac{1}{e+1}|z_s|^2})$$

steps.

*Claim 6. Let*

$$SAT(s) = \{e' : \exists t \leq s \ (\Re_{e'} \textit{ is satisfied at stage } t)\}.$$

*For any $k \geq 1$ there is a procedure which computes $A(z_s)$, $SAT(s)$ and $b_{e'}(s)$ for $k < e' \leq s$ in $O(2^{\frac{1}{k}|z_s|^2})$ steps.*

*Proof.* Fix $k \geq 1$ and, by Claim 3, fix $s_0$ such that no requirement $\Re_{e'}$ with $e' \leq k$ is active after stage $s_0$. It suffices to give a procedure which, for $s > s_0$, computes $A(z_s)$, $SAT(s)$ and $b_{e'}(s)$ (for $k < e' \leq s$) from $A \upharpoonright z_s$, $SAT(s-1)$ and $b_{e'}(s-1)$ (for $k < e' \leq s-1$) in $O(2^{\frac{1}{k+1}|z_s|^2})$ steps (where $SAT(-1) = \emptyset$). Then the claim follows by induction.

Now, given $s > s_0$, $A \upharpoonright z_s$, $SAT(s-1)$ and $b_e(s-1)$ (for $k < e' \leq s-1$), we proceed as follows.

- Compute $r_s$.

  *This can be done in $O(2^{8 \cdot |z_s|})$ steps.*

- Decide whether there is a requirement $\Re_e$, $e < |z_s|$ which is eligible to act at stage $s$ and, if so, compute $e$, the least $x$ as in (17), and $i = 1 - E_{e_0}(x)$.

  *To do so, for any $e < |z_s|$, such that $k < e$ and $e \notin SAT(s-1)$, it suffices to check whether (17) holds and, if so, to compute the least witness $x$ for (17) and to decide whether $x \in E_{e_0}$. By Claim 5 this can be done in $O(s \cdot 2^{\frac{1}{k+2}|z_s|^2}) = O(2^{\frac{1}{k+2}|z_s|^2+|z_s|})$ steps.*

- If no requirement is eligible to act at stage $s$ then $A(z_s) = r_s$, $SAT(s) = SAT(s-1)$ and $b_e(s) = b_e(s-1)$ for all $e \leq s$.

- If $\Re_e$ is eligible to act at stage $s$ then check whether (19) holds. If so, $A(z_s) = r_s$, $SAT(s) = SAT(s-1)$ and $b_e(s)$ can be computed from (20); otherwise, $A(z_s) = i$, $SAT(s) = SAT(s-1) \cup \{e\}$, and $b_e(s) = 0$. In either case, $b_{e'}(s) = b_{e'}(s-1)$ for $e' \neq e$.

  *This can be done in $O(2^{8 \cdot |z_s|})$ steps.*

This completes the procedure. By the analysis of the time required for performing the individual steps, the procedure can be completed in time $O(2^{\frac{1}{k+1}|z_s|^2})$.

*Claim 7. A satisfies* (9).

*Proof.* This is immediate by Claim 6.

*Claim 8.* $A \in \mathrm{EXP}$ *and A is* E-*trivial.*

*Proof.* The former is immediate by Claim 7 while, as shown above, the latter follows from Claims 4 and 7.

This completes the proof of Lemma 5.2. □

# 6 Conclusion

We can summarize the relations among the weak hardness notions for E and EXP in the following theorem.

**Theorem 6.1** *For any set $A$ the following hold.*

$$\begin{array}{ccc} A \text{ E-}\textit{hard} & \Leftrightarrow & A \text{ EXP-}\textit{hard} \\ \Downarrow & & \Downarrow \\ A \text{ E-}\textit{measure hard} & \Rightarrow & A \text{ EXP-}\textit{measure hard} \\ \Downarrow & & \Downarrow \\ A \textit{ strongly } \text{E-}\textit{nontrivial} & \Rightarrow & A \textit{ strongly } \text{EXP-}\textit{nontrivial} \\ \Downarrow & & \Downarrow \\ A \text{ E-}\textit{nontrivial} & & A \text{ EXP-}\textit{nontrivial} \end{array} \tag{24}$$

*Moreover, (up to transitive closure) no other implications hold and sets witnessing the failure of the other relations can be found in* E.

Proof. The positive relations in (24) hold by Lemma 3.1, by the coincidence of hardness for E and EXP, by Theorem 3.3, and by Lemma 3.4.

To show that no other relations hold and that the failure is witnessed by sets in E, we first observe that, by Theorem 3.2, none of the weak hardness concepts for E *and* EXP implies any of the stronger (weak) hardness concepts for E *or* EXP. So it only remains to argue that E-nontriviality does not imply EXP-nontriviality and that none of the weak hardness notions for EXP implies any of the weak hardness notions for E. But the former is true by Theorem 4.1, while the latter is true by Theorem 5.1 which shows that the strongest weak hardness notion for EXP does not imply the weakest weak hardness notion for E. □

Diagram (24) can be extended by including Ambos-Spies's weak hardness notion based on resource-bounded Baire category. In fact, in [1] weak hardness notions were introduced for various time-bounded category concepts. Here we refer to the category concept called AFH-category there and call the corresponding weak hardness notion category hardness. AFH-category proved to be useful for analysing time-bounded measure (see [4] and [5] for more details) since - in contrast to the classical Baire category concept - this concept is compatible with measure. As shown in [1], category hardness for E (EXP) is a proper generalization of measure hardness for E (EXP) and, as shown in [2], strong nontriviality for E (EXP) is a proper generalization of category hardness for E (EXP). Moreover, in [5] the analog of Theorem 3.3 for category hardness has been shown. By these results and by Theorem 6.1, for any set $A$,

$$\begin{array}{ccc} A \text{ E-hard} & \Leftrightarrow & A \text{ EXP-hard} \\ \Downarrow & & \Downarrow \\ A \text{ E-measure hard} & \Rightarrow & A \text{ EXP-measure hard} \\ \Downarrow & & \Downarrow \\ A \text{ E-category hard} & \Rightarrow & A \text{ EXP-category hard} \\ \Downarrow & & \Downarrow \\ A \text{ strongly E-nontrivial} & \Rightarrow & A \text{ strongly EXP-nontrivial} \\ \Downarrow & & \Downarrow \\ A \text{ E-nontrivial} & & A \text{ EXP-nontrivial} \end{array}$$

holds. Moreover, no other implications hold in general and sets witnessing the failure of the other relations can be found in E.

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