HOW MUCH DOES A TYPICAL SET KNOW ABOUT EXPONENTIAL TIME?

Klaus Ambos-Spies and Timur Bakibayev

Ruprecht-Karls-Universität (Heidelberg)
Al-Farabi Kazakh National University (Almaty)

Workshop on Computability Theory
Barcelona, 17 July 2011

Overview

- Basic concepts from complexity theory
  - The exponential time classes E and EXP
  - Resource-bounded measure and pseudo-randomness (what are typical sets?)
  - Weak hardness notions for exponential time (some views on the nonnegligible parts of E)
- Are typical sets weakly hard for E?
  - Typical sets among all sets
  - Typical sets in E
  - Typical sets in EXP
  - Typical elementarily recursive sets (i.e., sets in EL)
  - Typical computable sets (i.e., sets in REC)

A (somewhat vague) analogy

A (somewhat vague) analogy

REC	P = PTIME
(theoretically) solvable problems	(feasibly solv. =) tractable problems

A (somewhat vague) analogy

REC	$\mathrm{P} = \mathrm{PTIME}$
(theoretically) solvable problems	(feasibly solv. =) tractable problems
$\leq_{\mathrm{T}}$ $(\leq_{\mathrm{m}})$	$\leq^{\mathrm{P}}_{\mathrm{m}}$ $(\leq^{\mathrm{P}}_{\mathrm{T}})$
effective reductions	feasible (=poly-time) reductions

A (somewhat vague) analogy

REC (theoretically) solvable problems	$\mathrm{P} = \mathrm{PTIME}$ (feasibly solv. =) tractable problems
$\leq_{\mathrm{T}}$ $(\leq_{\mathrm{m}})$ effective reductions	$\leq^{\mathrm{P}}_{\mathrm{m}}$ $(\leq^{\mathrm{P}}_{\mathrm{T}})$ feasible (=poly-time) reductions
$\mathrm{REC} \subset \mathrm{CE} \subset \Delta^0_2$	$\mathrm{P} \subset \mathrm{DT}(2^{O(n)}) \subset \mathrm{DT}(2^{poly(n)})$

A (somewhat vague) analogy

REC	$\mathrm{P} = \mathrm{PTIME}$
(theoretically) solvable problems	(feasibly solv. =) tractable problems
$\leq_{\mathrm{T}}$ $(\leq_{\mathrm{m}})$	$\leq^{\mathrm{P}}_{\mathrm{m}}$ $(\leq^{\mathrm{P}}_{\mathrm{T}})$
effective reductions	feasible (=poly-time) reductions
$\mathrm{REC} \subset \mathrm{CE} \subset \Delta^0_2$	$\mathrm{P} \subset \mathrm{DT}(2^{O(n)}) \subset \mathrm{DT}(2^{poly(n)})$
CE-hard $\Rightarrow$ noncomputable	$\mathrm{DT}(2^{O(n)})$-hard $\Rightarrow$ intractable

A (somewhat vague) analogy

REC (theoretically) solvable problems	$\mathrm{P} = \mathrm{PTIME}$ (feasibly solv. =) tractable problems
$\leq_{\mathrm{T}}$ $(\leq_{\mathrm{m}})$ effective reductions	$\leq_{\mathrm{m}}^{\mathrm{P}}$ $(\leq_{\mathrm{T}}^{\mathrm{P}})$ feasible (=poly-time) reductions
$\mathrm{REC} \subset \mathrm{CE} \subset \Delta_2^0$	$\mathrm{P} \subset \mathrm{DT}(2^{O(n)}) \subset \mathrm{DT}(2^{poly(n)})$
CE-hard $\Rightarrow$ noncomputable	$\mathrm{DT}(2^{O(n)})$-hard $\Rightarrow$ intractable
$\mathrm{CE}_{\leq_{\mathrm{T}}} = \Delta_2^0$	$\mathrm{DT}(2^{O(n)})_{\leq_{\mathrm{m}}^{\mathrm{P}}} = \mathrm{DT}(2^{poly(n)})$

A (somewhat vague) analogy

REC (theoretically) solvable problems	$\mathrm{P} = \mathrm{PTIME}$ (feasibly solv. =) tractable problems
$\leq_{\mathrm{T}}$ $(\leq_{\mathrm{m}})$ effective reductions	$\leq^{\mathrm{P}}_{\mathrm{m}}$ $(\leq^{\mathrm{P}}_{\mathrm{T}})$ feasible (=poly-time) reductions
$\mathrm{REC} \subset \mathrm{CE} \subset \Delta^0_2$	$\mathrm{P} \subset \mathrm{DT}(2^{O(n)}) \subset \mathrm{DT}(2^{poly(n)})$
CE-hard $\Rightarrow$ noncomputable	$\mathrm{DT}(2^{O(n)})$-hard $\Rightarrow$ intractable
$\mathrm{CE}_{\leq_{\mathrm{T}}} = \Delta^0_2$	$\mathrm{DT}(2^{O(n)})_{\leq^{\mathrm{P}}_{\mathrm{m}}} = \mathrm{DT}(2^{poly(n)})$

In complexity theory we consider sets of binary strings and measure the complexity in the length of the strings. We may identify the number n with the nth string z_n in the canoncial ordering (NB: $|n| := |z_n| \approx \log(n)$).

The exponential-time classes: definitions

The exponential-time classes: definitions

- Linear Exponential Time:

$$\mathrm{E} = \mathrm{DTIME}(2^{lin}) = \cup_{k \geq 1} \mathrm{E}_k$$

where

$$\mathrm{E}_k = \mathrm{DTIME}(2^{kn})$$

The exponential-time classes: definitions

- Linear Exponential Time:

$$\mathrm{E} = \mathrm{DTIME}(2^{lin}) = \cup_{k \geq 1} \mathrm{E}_k$$

where

$$\mathrm{E}_k = \mathrm{DTIME}(2^{kn})$$

- Polynomial Exponential Time:

$$\mathrm{EXP} = \mathrm{DTIME}(2^{poly}) = \cup_{k \geq 1} \mathrm{EXP}_k$$

where

$$\mathrm{EXP}_k = \mathrm{DTIME}(2^{n^k})$$

The exponential-time classes: definitions

- Linear Exponential Time:

$$\mathrm{E} = \mathrm{DTIME}(2^{lin}) = \cup_{k \geq 1} \mathrm{E}_k$$

where

$$\mathrm{E}_k = \mathrm{DTIME}(2^{kn})$$

- Polynomial Exponential Time:

$$\mathrm{EXP} = \mathrm{DTIME}(2^{poly}) = \cup_{k \geq 1} \mathrm{EXP}_k$$

where

$$\mathrm{EXP}_k = \mathrm{DTIME}(2^{n^k})$$

By the time hierachy theorem:

$$\mathrm{P} \subset \mathrm{E}_1 (= \mathrm{EXP}_1) \subset \mathrm{E}_2 \subset \mathrm{E}_3 \cdots \subset \mathrm{E}$$

$$\mathrm{EXP}_1 \subset \mathrm{E} \subset \mathrm{EXP}_2 \subset \mathrm{EXP}_3 \subset \cdots \subset \mathrm{EXP} \subset \mathrm{REC}$$

The exponential-time classes: some facts

The exponential-time classes: some facts

- E is generally considered to be the least natural (i.e. sufficiently closed) time complexity class which allows diagonalizations over polynomial-time concepts (so, in particular, is a proper super class of P).

The exponential-time classes: some facts

- E is generally considered to be the least natural (i.e. sufficiently closed) time complexity class which allows diagonalizations over polynomial-time concepts (so, in particular, is a proper super class of P).

  The standard way to prove intractability of a problem A is to show that A is E-hard.

The exponential-time classes: some facts

- E is generally considered to be the least natural (i.e. sufficiently closed) time complexity class which allows diagonalizations over polynomial-time concepts (so, in particular, is a proper super class of P).

  The standard way to prove intractability of a problem A is to show that A is E-hard.

- EXP is the least (deterministic) time complexity class which is known to contain NP.

The exponential-time classes: some facts

- E is generally considered to be the least natural (i.e. sufficiently closed) time complexity class which allows diagonalizations over polynomial-time concepts (so, in particular, is a proper super class of P).

  The standard way to prove intractability of a problem A is to show that A is E-hard.

- EXP is the least (deterministic) time complexity class which is known to contain NP.

- For any set $A \in \mathrm{EXP}_k$ there is a set $A' \in \mathrm{E}_1$ such that $A =^p_m A'$:

The exponential-time classes: some facts

- E is generally considered to be the least natural (i.e. sufficiently closed) time complexity class which allows diagonalizations over polynomial-time concepts (so, in particular, is a proper super class of P).

  The standard way to prove intractability of a problem A is to show that A is E-hard.

- EXP is the least (deterministic) time complexity class which is known to contain NP.

- For any set $A \in \mathrm{EXP}_k$ there is a set $A' \in \mathrm{E}_1$ such that $A =^p_m A'$:

  PADDING LEMMA. For $A \in \mathrm{EXP}_k$, $A_k = \{0^{|x|^k} x : x \in A\} \in \mathrm{E}_1$.

The exponential-time classes: some facts

- E is generally considered to be the least natural (i.e. sufficiently closed) time complexity class which allows diagonalizations over polynomial-time concepts (so, in particular, is a proper super class of P).

  The standard way to prove intractability of a problem A is to show that A is E-hard.

- EXP is the least (deterministic) time complexity class which is known to contain NP.

- For any set $A \in \mathrm{EXP}_k$ there is a set $A' \in \mathrm{E}_1$ such that $A =^p_m A'$:

  PADDING LEMMA. For $A \in \mathrm{EXP}_k$, $A_k = \{0^{|x|^k} x : x \in A\} \in \mathrm{E}_1$.

  So E is not downward-closed under P-m-reducibility and (as one can easily check) EXP is the downward-closure of E under P-m-reducibility.

The exponential-time classes: some facts

- E is generally considered to be the least natural (i.e. sufficiently closed) time complexity class which allows diagonalizations over polynomial-time concepts (so, in particular, is a proper super class of P).

  The standard way to prove intractability of a problem A is to show that A is E-hard.

- EXP is the least (deterministic) time complexity class which is known to contain NP.

- For any set $A \in \mathrm{EXP}_k$ there is a set $A' \in \mathrm{E}_1$ such that $A =^p_m A'$:

  PADDING LEMMA. For $A \in \mathrm{EXP}_k$, $A_k = \{0^{|x|^k} x : x \in A\} \in \mathrm{E}_1$.

  So E is not downward-closed under P-m-reducibility and (as one can easily check) EXP is the downward-closure of E under P-m-reducibility. So, in particular, hardness for E and EXP coincide.

Resource-bounded measure and pseudo-randomness

Resource-bounded measure and pseudo-randomness

Definition (Martingales and time-bounded martingales)

Resource-bounded measure and pseudo-randomness

Definition (Martingales and time-bounded martingales)

- A (normed) martingale is a real valued function $d : \{0,1\}^* \to [0,\infty)$ such that $d(\lambda) = 1$ and, for every $x \in \{0,1\}^*$, the following equality holds.

$$\frac{d(x0) + d(x1)}{2} = d(x) \quad \text{(fairness condition)} \tag{1}$$

Resource-bounded measure and pseudo-randomness

Definition (Martingales and time-bounded martingales)

- A (normed) martingale is a real valued function $d : \{0,1\}^* \to [0,\infty)$ such that $d(\lambda) = 1$ and, for every $x \in \{0,1\}^*$, the following equality holds.

$$\frac{d(x0) + d(x1)}{2} = d(x) \quad \text{(fairness condition)} \tag{1}$$

- A martingale d succeeds on a set A if $\limsup_{n \geq 0} d(A \upharpoonright n) = \infty$. A martingale d succeeds on a class C if it succeeds on all sets $A \in \mathrm{C}$.

Resource-bounded measure and pseudo-randomness

Definition (Martingales and time-bounded martingales)

- A (normed) martingale is a real valued function $d : \{0,1\}^* \to [0,\infty)$ such that $d(\lambda) = 1$ and, for every $x \in \{0,1\}^*$, the following equality holds.

$$\frac{d(x0) + d(x1)}{2} = d(x) \quad \text{(fairness condition)} \tag{1}$$

- A martingale d succeeds on a set A if $\limsup_{n \geq 0} d(A \upharpoonright n) = \infty$. A martingale d succeeds on a class C if it succeeds on all sets $A \in \mathrm{C}$.
- The (betting) strategy s_d underlying the martingale d is the function

$$s_d(x) = \begin{cases} \frac{d(x0)}{2d(x)} & \text{if } d(x) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Resource-bounded measure and pseudo-randomness

Definition (Martingales and time-bounded martingales)

- A (normed) martingale is a real valued function $d : \{0,1\}^* \to [0, \infty)$ such that $d(\lambda) = 1$ and, for every $x \in \{0,1\}^*$, the following equality holds.

$$\frac{d(x0) + d(x1)}{2} = d(x) \quad \text{(fairness condition)} \tag{1}$$

- A martingale d succeeds on a set A if $\limsup_{n \geq 0} d(A \upharpoonright n) = \infty$. A martingale d succeeds on a class C if it succeeds on all sets $A \in \mathrm{C}$.
- The (betting) strategy s_d underlying the martingale d is the function

$$s_d(x) = \begin{cases} \frac{d(x0)}{2d(x)} & \text{if } d(x) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

- A $t(n)$-martingale d is a rational valued martingale $d : \{0,1\}^* \to \mathbb{Q} \cap [0, \infty)$ such that, for the underlying strategy s_d, $s_d \in \mathrm{DTIME}(t(n))$.

Resource-bounded measure and pseudo-randomness

Resource-bounded measure and pseudo-randomness

Definition (Measure, time-bd. measure, pseudo-randomness)

- A class C has measure 0 if there is a martingale which succeeds on C; and C has measure 1 if the complement $\overline{\mathrm{C}}$ of C has measure 0.

Resource-bounded measure and pseudo-randomness

Definition (Measure, time-bd. measure, pseudo-randomness)

- A class C has measure 0 if there is a martingale which succeeds on C; and C has measure 1 if the complement $\overline{\mathrm{C}}$ of C has measure 0.
- A class C has $t(n)$-measure 0 if there is a $t(n)$-martingale which succeeds on C; and C has $t(n)$-measure 1 if the complement $\overline{\mathrm{C}}$ of C has $t(n)$-measure 0.

Resource-bounded measure and pseudo-randomness

Definition (Measure, time-bd. measure, pseudo-randomness)

- A class C has measure 0 if there is a martingale which succeeds on C; and C has measure 1 if the complement $\overline{\mathrm{C}}$ of C has measure 0.
- A class C has $t(n)$-measure 0 if there is a $t(n)$-martingale which succeeds on C; and C has $t(n)$-measure 1 if the complement $\overline{\mathrm{C}}$ of C has $t(n)$-measure 0.
- A set A is $t(n)$-random if no $t(n)$-martingale succeeds on A (i.e., if the singleton $\{A\}$ does not have $t(n)$-measure 0).

Resource-bounded measure and pseudo-randomness

Definition (Measure, time-bd. measure, pseudo-randomness)

- A class C has measure 0 if there is a martingale which succeeds on C; and C has measure 1 if the complement $\overline{\mathrm{C}}$ of C has measure 0.
- A class C has $t(n)$-measure 0 if there is a $t(n)$-martingale which succeeds on C; and C has $t(n)$-measure 1 if the complement $\overline{\mathrm{C}}$ of C has $t(n)$-measure 0.
- A set A is $t(n)$-random if no $t(n)$-martingale succeeds on A (i.e., if the singleton $\{A\}$ does not have $t(n)$-measure 0).

As Lutz has shown one can define (pseudo) measures on complexity classes by considering families of martingales of corresponding complexity:

Resource-bounded measure and pseudo-randomness

Definition (Measure, time-bd. measure, pseudo-randomness)

- A class C has measure 0 if there is a martingale which succeeds on C; and C has measure 1 if the complement $\overline{\mathrm{C}}$ of C has measure 0.
- A class C has $t(n)$-measure 0 if there is a $t(n)$-martingale which succeeds on C; and C has $t(n)$-measure 1 if the complement $\overline{\mathrm{C}}$ of C has $t(n)$-measure 0.
- A set A is $t(n)$-random if no $t(n)$-martingale succeeds on A (i.e., if the singleton $\{A\}$ does not have $t(n)$-measure 0).

As Lutz has shown one can define (pseudo) measures on complexity classes by considering families of martingales of corresponding complexity:

E	p-measure: n^k-martingales ($k \geq 1$)

Resource-bounded measure and pseudo-randomness

Definition (Measure, time-bd. measure, pseudo-randomness)

- A class C has measure 0 if there is a martingale which succeeds on C; and C has measure 1 if the complement $\overline{\mathrm{C}}$ of C has measure 0.
- A class C has $t(n)$-measure 0 if there is a $t(n)$-martingale which succeeds on C; and C has $t(n)$-measure 1 if the complement $\overline{\mathrm{C}}$ of C has $t(n)$-measure 0.
- A set A is $t(n)$-random if no $t(n)$-martingale succeeds on A (i.e., if the singleton $\{A\}$ does not have $t(n)$-measure 0).

As Lutz has shown one can define (pseudo) measures on complexity classes by considering families of martingales of corresponding complexity:

E	p-measure: n^k-martingales $(k \geq 1)$
EXP	p_2-measure: $2^{(\log n)^k}$-martingales $(k \geq 1)$

Resource-bounded measure and pseudo-randomness

Definition (Measure, time-bd. measure, pseudo-randomness)

- A class C has measure 0 if there is a martingale which succeeds on C; and C has measure 1 if the complement $\overline{\mathrm{C}}$ of C has measure 0.
- A class C has $t(n)$-measure 0 if there is a $t(n)$-martingale which succeeds on C; and C has $t(n)$-measure 1 if the complement $\overline{\mathrm{C}}$ of C has $t(n)$-measure 0.
- A set A is $t(n)$-random if no $t(n)$-martingale succeeds on A (i.e., if the singleton $\{A\}$ does not have $t(n)$-measure 0).

As Lutz has shown one can define (pseudo) measures on complexity classes by considering families of martingales of corresponding complexity:

E	p-measure: n^k-martingales ($k \geq 1$)
EXP	p_2-measure: $2^{(\log n)^k}$-martingales ($k \geq 1$)
EL	*elementary*-measure: elementary-recursive martingales

Resource-bounded measure and pseudo-randomness

Definition (Measure, time-bd. measure, pseudo-randomness)

- A class C has measure 0 if there is a martingale which succeeds on C; and C has measure 1 if the complement $\overline{\mathrm{C}}$ of C has measure 0.
- A class C has $t(n)$-measure 0 if there is a $t(n)$-martingale which succeeds on C; and C has $t(n)$-measure 1 if the complement $\overline{\mathrm{C}}$ of C has $t(n)$-measure 0.
- A set A is $t(n)$-random if no $t(n)$-martingale succeeds on A (i.e., if the singleton $\{A\}$ does not have $t(n)$-measure 0).

As Lutz has shown one can define (pseudo) measures on complexity classes by considering families of martingales of corresponding complexity:

E	p-measure: n^k-martingales ($k \geq 1$)
EXP	p_2-measure: $2^{(\log n)^k}$-martingales ($k \geq 1$)
EL	*elementary*-measure: elementary-recursive martingales
REC	*computable*-measure: computable martingales

Resource-bounded measure and pseudo-randomness

Resource-bounded measure and pseudo-randomness

Definition (p-Measure and measure for E)

Resource-bounded measure and pseudo-randomness

Definition (p-Measure and measure for E)

- A class C has p-measure 0 if C has n^k-measure 0 for some $k \geq 1$; and C has p-measure 1 if $\overline{\mathrm{C}}$ has p-measure 0.

Resource-bounded measure and pseudo-randomness

Definition (p-Measure and measure for E)

- A class C has p-measure 0 if C has n^k-measure 0 for some $k \geq 1$; and C has p-measure 1 if $\overline{\mathrm{C}}$ has p-measure 0.
- A class C has measure 0 in E if $\mathrm{C} \cap \mathrm{E}$ has p-measure 0; and C has measure 1 in E if the complement $\overline{\mathrm{C}}$ of C has measure 0 in E.

Resource-bounded measure and pseudo-randomness

Definition (p-Measure and measure for E)

- A class C has p-measure 0 if C has n^k-measure 0 for some $k \geq 1$; and C has p-measure 1 if $\overline{\mathrm{C}}$ has p-measure 0.
- A class C has measure 0 in E if $\mathrm{C} \cap \mathrm{E}$ has p-measure 0; and C has measure 1 in E if the complement $\overline{\mathrm{C}}$ of C has measure 0 in E.
- A set A is *p-random* if A is n^k-random for all $k \geq 1$ and A.

Resource-bounded measure and pseudo-randomness

Definition (p-Measure and measure for E)

- A class C has p-measure 0 if C has n^k-measure 0 for some $k \geq 1$; and C has p-measure 1 if $\overline{\mathrm{C}}$ has p-measure 0.
- A class C has measure 0 in E if $\mathrm{C} \cap \mathrm{E}$ has p-measure 0; and C has measure 1 in E if the complement $\overline{\mathrm{C}}$ of C has measure 0 in E.
- A set A is *p-random* if A is n^k-random for all $k \geq 1$ and A.

Lutzn has shown that the measure on E is consistent:

Resource-bounded measure and pseudo-randomness

Definition (p-Measure and measure for E)

- A class C has p-measure 0 if C has n^k-measure 0 for some $k \geq 1$; and C has p-measure 1 if $\overline{\mathrm{C}}$ has p-measure 0.
- A class C has measure 0 in E if $\mathrm{C} \cap \mathrm{E}$ has p-measure 0; and C has measure 1 in E if the complement $\overline{\mathrm{C}}$ of C has measure 0 in E.
- A set A is *p-random* if A is n^k-random for all $k \geq 1$ and A.

Lutzn has shown that the measure on E is consistent:

- E does not have p-measure 0, hence not measure 0 in E.

Resource-bounded measure and pseudo-randomness

Definition (p-Measure and measure for E)

- A class C has p-measure 0 if C has n^k-measure 0 for some $k \geq 1$; and C has p-measure 1 if $\overline{\mathrm{C}}$ has p-measure 0.
- A class C has measure 0 in E if $\mathrm{C} \cap \mathrm{E}$ has p-measure 0; and C has measure 1 in E if the complement $\overline{\mathrm{C}}$ of C has measure 0 in E.
- A set A is *p-random* if A is n^k-random for all $k \geq 1$ and A.

Lutzn has shown that the measure on E is consistent:

- E does not have p-measure 0, hence not measure 0 in E.
- For $k \geq 1$, E_k has p-measure 0, hence measure 0 in E.

Resource-bounded measure and pseudo-randomness

Definition (p-Measure and measure for E)

- A class C has *p*-measure 0 if C has n^k-measure 0 for some $k \geq 1$; and C has *p*-measure 1 if $\overline{\mathrm{C}}$ has p-measure 0.
- A class C has measure 0 in E if $\mathrm{C} \cap \mathrm{E}$ has p-measure 0; and C has measure 1 in E if the complement $\overline{\mathrm{C}}$ of C has measure 0 in E.
- A set A is *p-random* if A is n^k-random for all $k \geq 1$ and A.

Lutzn has shown that the measure on E is consistent:

- E does not have p-measure 0, hence not measure 0 in E.
- For $k \geq 1$, E_k has p-measure 0, hence measure 0 in E.
- There is no p-random set in E and no n^k-random set in E_k (but there are p-random sets in EXP_2 and n^k-random sets in E_{k+3}).

Resource-bounded measure and pseudo-randomness

Definition (p-Measure and measure for E)

- A class C has p-measure 0 if C has n^k-measure 0 for some $k \geq 1$; and C has p-measure 1 if $\overline{\mathrm{C}}$ has p-measure 0.
- A class C has measure 0 in E if $\mathrm{C} \cap \mathrm{E}$ has p-measure 0; and C has measure 1 in E if the complement $\overline{\mathrm{C}}$ of C has measure 0 in E.
- A set A is *p-random* if A is n^k-random for all $k \geq 1$ and A.

Lutzn has shown that the measure on E is consistent:

- E does not have p-measure 0, hence not measure 0 in E.
- For $k \geq 1$, E_k has p-measure 0, hence measure 0 in E.
- There is no p-random set in E and no n^k-random set in E_k (but there are p-random sets in EXP_2 and n^k-random sets in E_{k+3}).
- A class C has measure 0 in E if and only if there is a number k such that $\mathrm{C} \cap \mathrm{E}$ does not contain any n^k-random set.

Typicalness

Typicalness

- Measures on EXP, EL and REC are defined correspondingly, and there are corresponding characterizations in terms of time bounded randomness.

Typicalness

- Measures on EXP, EL and REC are defined correspondingly, and there are corresponding characterizations in terms of time bounded randomness.

  A set A is computably random or *rec*-random if it is $t(n)$-random for all computable functions t. Note that no *rec*-random set is computable while there are computable $t(n)$-random sets for all computable functions $t(n)$.

Typicalness

- Measures on EXP, EL and REC are defined correspondingly, and there are corresponding characterizations in terms of time bounded randomness.

  A set A is computably random or *rec*-random if it is $t(n)$-random for all computable functions t. Note that no *rec*-random set is computable while there are computable $t(n)$-random sets for all computable functions $t(n)$.

- Now we can define that a property $\mathcal{P}$ is

Typicalness

- Measures on EXP, EL and REC are defined correspondingly, and there are corresponding characterizations in terms of time bounded randomness.

  A set A is computably random or *rec*-random if it is $t(n)$-random for all computable functions t. Note that no *rec*-random set is computable while there are computable $t(n)$-random sets for all computable functions $t(n)$.

- Now we can define that a property $\mathcal{P}$ is
  - typical (untypical) for E if $\{A \in \mathrm{E} : \mathcal{P}(A)\}$ has measure 1 (0) in E

Typicalness

- Measures on EXP, EL and REC are defined correspondingly, and there are corresponding characterizations in terms of time bounded randomness.

  A set A is computably random or *rec*-random if it is $t(n)$-random for all computable functions t. Note that no *rec*-random set is computable while there are computable $t(n)$-random sets for all computable functions $t(n)$.

- Now we can define that a property $\mathcal{P}$ is
  - typical (untypical) for E if $\{A \in \mathrm{E} : \mathcal{P}(A)\}$ has measure 1 (0) in E
  - typical (untypical) for EXP if $\{A \in \mathrm{EXP} : \mathcal{P}(A)\}$ has measure 1 (0) in EXP

Typicalness

- Measures on EXP, EL and REC are defined correspondingly, and there are corresponding characterizations in terms of time bounded randomness.

  A set A is computably random or *rec*-random if it is $t(n)$-random for all computable functions t. Note that no *rec*-random set is computable while there are computable $t(n)$-random sets for all computable functions $t(n)$.

- Now we can define that a property $\mathcal{P}$ is
  - typical (untypical) for E if $\{A \in \mathrm{E} : \mathcal{P}(A)\}$ has measure 1 (0) in E
  - typical (untypical) for EXP if $\{A \in \mathrm{EXP} : \mathcal{P}(A)\}$ has measure 1 (0) in EXP
  - typical (untypical) for EL if $\{A \in \mathrm{EL} : \mathcal{P}\}$ has measure 1 (0) in EL

Typicalness

- Measures on EXP, EL and REC are defined correspondingly, and there are corresponding characterizations in terms of time bounded randomness.

  A set A is computably random or *rec*-random if it is $t(n)$-random for all computable functions t. Note that no *rec*-random set is computable while there are computable $t(n)$-random sets for all computable functions $t(n)$.

- Now we can define that a property $\mathcal{P}$ is
  - typical (untypical) for E if $\{A \in \mathrm{E} : \mathcal{P}(A)\}$ has measure 1 (0) in E
  - typical (untypical) for EXP if $\{A \in \mathrm{EXP} : \mathcal{P}(A)\}$ has measure 1 (0) in EXP
  - typical (untypical) for EL if $\{A \in \mathrm{EL} : \mathcal{P}\}$ has measure 1 (0) in EL
  - typical (untypical) for the computable sets if $\{A \in \mathrm{REC} : \mathcal{P}(A)\}$ has measure 1 (0) in REC

Typicalness

- Measures on EXP, EL and REC are defined correspondingly, and there are corresponding characterizations in terms of time bounded randomness.

  A set A is computably random or *rec*-random if it is $t(n)$-random for all computable functions t. Note that no *rec*-random set is computable while there are computable $t(n)$-random sets for all computable functions $t(n)$.

- Now we can define that a property $\mathcal{P}$ is
  - typical (untypical) for E if $\{A \in \mathrm{E} : \mathcal{P}(A)\}$ has measure 1 (0) in E
  - typical (untypical) for EXP if $\{A \in \mathrm{EXP} : \mathcal{P}(A)\}$ has measure 1 (0) in EXP
  - typical (untypical) for EL if $\{A \in \mathrm{EL} : \mathcal{P}\}$ has measure 1 (0) in EL
  - typical (untypical) for the computable sets if $\{A \in \mathrm{REC} : \mathcal{P}(A)\}$ has measure 1 (0) in REC
  - typical (untypical) for the sets in general if $\{A : \mathcal{P}(A)\}$ has (classical) measure 1 (0)

Fractions of $\mathbb{E}$: hardness and weak hardness

Fractions of E: hardness and weak hardness

- We now want to quantify the amount of knowledge a set A has about E.

Fractions of E: hardness and weak hardness

- We now want to quantify the amount of knowledge a set A has about E.Obviously this is determined by the "size" of the part of E which can be reduced to A, i.e., by the size of

$$\mathrm{P_m}(A) \cap \mathrm{E} = \{B \in \mathrm{E} : B \leq^{\mathrm{P}}_{\mathrm{m}} A\}$$

Fractions of E: hardness and weak hardness

- We now want to quantify the amount of knowledge a set A has about E.Obviously this is determined by the "size" of the part of E which can be reduced to A, i.e., by the size of

$$\mathrm{P_m}(A) \cap \mathrm{E} = \{B \in \mathrm{E} : B \leq_{\mathrm{m}}^{\mathrm{P}} A\}$$

- The extremes are:

Fractions of E: hardness and weak hardness

- We now want to quantify the amount of knowledge a set A has about E.Obviously this is determined by the “size” of the part of E which can be reduced to A, i.e., by the size of

$$\mathrm{P_m}(A) \cap \mathrm{E} = \{B \in \mathrm{E} : B \leq^{\mathrm{P}}_{\mathrm{m}} A\}$$

- The extremes are:
  - A is E-hard: $\mathrm{P_m}(A) \cap \mathrm{E} = \mathrm{E}$ (i.e., A knows all about E)

Fractions of E: hardness and weak hardness

- We now want to quantify the amount of knowledge a set A has about E.Obviously this is determined by the "size" of the part of E which can be reduced to A, i.e., by the size of

$$\mathrm{P_m}(A) \cap \mathrm{E} = \{B \in \mathrm{E} : B \leq^{\mathrm{P}}_{\mathrm{m}} A\}$$

- The extremes are:
  - A is E-hard: $\mathrm{P_m}(A) \cap \mathrm{E} = \mathrm{E}$ (i.e., A knows all about E)
  - A is E-useless: $\mathrm{P_m}(A) \cap \mathrm{E} = \mathrm{P}$ (i.e., A doesn't know anything about E) (In the following we say that A is E-useful if $\mathrm{P} \subset \mathrm{P_m}(A) \cap \mathrm{E}$.)

Fractions of E: hardness and weak hardness

- We now want to quantify the amount of knowledge a set A has about E.Obviously this is determined by the "size" of the part of E which can be reduced to A, i.e., by the size of

$$\mathrm{P_m}(A) \cap \mathrm{E} = \{B \in \mathrm{E} : B \leq^{\mathrm{P}}_{\mathrm{m}} A\}$$

- The extremes are:
  - A is E-hard: $\mathrm{P_m}(A) \cap \mathrm{E} = \mathrm{E}$ (i.e., A knows all about E)
  - A is E-useless: $\mathrm{P_m}(A) \cap \mathrm{E} = \mathrm{P}$ (i.e., A doesn't know anything about E)
    (In the following we say that A is E-useful if $\mathrm{P} \subset \mathrm{P_m}(A) \cap \mathrm{E}$.)

- Intermediate cases are provided by the weak hardness notions for E, a concept originally proposed by Lutz (1995).

Fractions of E: hardness and weak hardness

Fractions of E: hardness and weak hardness

- While a set is hard for a complexity class C if all sets in C can be reduced to A, Lutz proposed to call a set A weakly hard for C if a nonnegligible part of C can be reduced A.

Fractions of E: hardness and weak hardness

- While a set is hard for a complexity class C if all sets in C can be reduced to A, Lutz proposed to call a set A weakly hard for C if a nonnegligible part of C can be reduced A.

  Lutz further argued that a part of C is nonnegligible if it does not have measure 0 in C:

Fractions of E: hardness and weak hardness

- While a set is hard for a complexity class C if all sets in C can be reduced to A, Lutz proposed to call a set A weakly hard for C if a nonnegligible part of C can be reduced A.

  Lutz further argued that a part of C is nonnegligible if it does not have measure 0 in C:

  LUTZ (1995): A set A is measure-hard for E if $\mathrm{P_m}(A)$ does not have measure 0 in E, i.e., if $\mathrm{P_m}(A) \cap \mathrm{E}$ does not have p-measure 0.

Fractions of E: hardness and weak hardness

- While a set is hard for a complexity class C if all sets in C can be reduced to A, Lutz proposed to call a set A weakly hard for C if a nonnegligible part of C can be reduced A.

  Lutz further argued that a part of C is nonnegligible if it does not have measure 0 in C:

  LUTZ (1995): A set A is measure-hard for E if $\mathrm{P_m}(A)$ does not have measure 0 in E, i.e., if $\mathrm{P_m}(A) \cap \mathrm{E}$ does not have p-measure 0.

- One obtains alternative weak hardness notions by giving different interpretations of the nonnegligible parts of E. E.g. this has been done in terms of Baire category / genericity in place of measure / randomness (A-S 1996).

  Here we consider other interpretations in terms of complexity theory due to A-S und Bakibayev (ICALP 2010):

Fractions of E: hardness and weak hardness

Fractions of E: hardness and weak hardness

- A set A is strongly nontrivial for E if, for any $k \geq 1$, there is an almost everywhere 2^{kn}-complex set (i.e., a $\mathrm{DTIME}(2^{kn})$-bi-immune set) in E which can be reduced to A:

$$\forall\, k \geq 1\; \exists\, A_k \in \mathrm{E}\; (A_k\ \mathrm{E}_k\text{-bi-immune}\ \&\ A_k \leq^{\mathrm{P}}_{\mathrm{m}} A)$$

Fractions of E: hardness and weak hardness

- A set A is strongly nontrivial for E if, for any $k \geq 1$, there is an almost everywhere 2^{kn}-complex set (i.e., a $\mathrm{DTIME}(2^{kn})$-bi-immune set) in E which can be reduced to A:

$$\forall\, k \geq 1\, \exists\, A_k \in \mathrm{E}\ (A_k\ \mathrm{E}_k\text{-bi-immune}\ \&\ A_k \leq^{\mathrm{P}}_{\mathrm{m}} A)$$

- A set A is nontrivial for E if, for any $k \geq 1$, there is an infinitely-often 2^{kn}-complex set in E which can be reduced to A:

$$\forall\, k \geq 1\, \exists\, A_k \in \mathrm{E} \setminus\ E_k\ (A_k \leq^{\mathrm{P}}_{\mathrm{m}} A)$$

Fractions of E: hardness and weak hardness

- A set A is strongly nontrivial for E if, for any $k \geq 1$, there is an almost everywhere 2^{kn}-complex set (i.e., a $\mathrm{DTIME}(2^{kn})$-bi-immune set) in E which can be reduced to A:

$$\forall\, k \geq 1\, \exists\, A_k \in \mathrm{E}\, (A_k\ \mathrm{E}_k\text{-bi-immune } \&\ A_k \leq^{\mathrm{P}}_{\mathrm{m}} A)$$

- A set A is nontrivial for E if, for any $k \geq 1$, there is an infinitely-often 2^{kn}-complex set in E which can be reduced to A:

$$\forall\, k \geq 1\, \exists\, A_k \in \mathrm{E} \setminus\ E_k\ (A_k \leq^{\mathrm{P}}_{\mathrm{m}} A)$$

Note that

$$\text{hard} \Rightarrow \text{measure-hard} \Rightarrow \text{strongly nontrivial} \Rightarrow \text{nontrivial} \Rightarrow \text{useful}$$

holds. Moreover, all implications are strict (even on E).

What weak hardness notions are typical?

Our goal is to complete the following table by filling in the corresponding (resource-bounded) measures:

	E	EXP	EL	REC	ALL
hard	?	?	?	?	?
measure hard	?	?	?	?	?
strongly non-trivial	?	?	?	?	?
nontrivial	?	?	?	?	?
useful	?	?	?	?	?

What weak hardness notions are typical? - Some first observations

What weak hardness notions are typical? - Some first observations

Since any level E_k of E has measure 0 in E it follows in particular that P has measure 0 in E. So typical sets in E are not in P, hence useful.

What weak hardness notions are typical? - Some first observations

Since any level E_k of E has measure 0 in E it follows in particular that P has measure 0 in E. So typical sets in E are not in P, hence useful.

By the Padding Lemma, this implies usefulness of the typical sets in EXP too.

What weak hardness notions are typical? - Some first observations

Since any level E_k of E has measure 0 in E it follows in particular that P has measure 0 in E. So typical sets in E are not in P, hence useful.

By the Padding Lemma, this implies usefulness of the typical sets in EXP too.

In fact, by a similar padding argument (Book, see A-S 1987), one can show that any set $A \in \mathrm{EL} \setminus \mathrm{P}$ has a predecessor in $\mathrm{E} \setminus \mathrm{P}$. So typical sets in EL are useful too.

What weak hardness notions are typical? - Some first observations

Since any level E_k of E has measure 0 in E it follows in particular that P has measure 0 in E. So typical sets in E are not in P, hence useful.

By the Padding Lemma, this implies usefulness of the typical sets in EXP too.

In fact, by a similar padding argument (Book, see A-S 1987), one can show that any set $A \in \mathrm{EL} \setminus \mathrm{P}$ has a predecessor in $\mathrm{E} \setminus \mathrm{P}$. So typical sets in EL are useful too.

	E	EXP	EL	REC	ALL
hard	?	?	?	?	?
measure hard	?	?	?	?	?
strongly non-trivial	?	?	?	?	?
nontrivial	?	?	?	?	?
useful	1	1	1	?	?

What weak hardness notions are typical among all sets ?

What weak hardness notions are typical among all sets ?

COMPUTABILITY THEORY:

THEOREM (SACKS, 1965). For any set $A \notin \mathrm{REC}$, $\mu(\{B : A \leq_{\mathrm{T}} B\}) = 0$.

What weak hardness notions are typical among all sets ?

COMPUTABILITY THEORY:

THEOREM (SACKS, 1965). For any set $A \notin \mathrm{REC}$, $\mu(\{B : A \leq_{\mathrm{T}} B\}) = 0$.

COMPLEXITY THEORY:

THEOREM (A-S, 1986). For any set $A \notin \mathrm{P}$, $\mu(\{B : A \leq_{\mathrm{m}}^{\mathrm{P}} B\}) = 0$.

What weak hardness notions are typical among all sets ?

COMPUTABILITY THEORY:

THEOREM (SACKS, 1965). For any set $A \notin \mathrm{REC}$, $\mu(\{B : A \leq_{\mathrm{T}} B\}) = 0$.

COMPLEXITY THEORY:

THEOREM (A-S, 1986). For any set $A \notin \mathrm{P}$, $\mu(\{B : A \leq_{\mathrm{m}}^{\mathrm{P}} B\}) = 0$.

By countable additivity of μ this implies:

COROLLARY. $\mu(\{A : \mathrm{P} \subset \mathrm{P_m}(A) \cap \mathrm{REC}\}) = \mu(\{A : \mathrm{P} \subset \mathrm{P_m}(A) \cap \mathrm{E}\}) = 0$. I.e., the class of E-useful sets has measure 0.

What weak hardness notions are typical among all sets ?

COMPUTABILITY THEORY:

THEOREM (SACKS, 1965). For any set $A \notin \mathrm{REC}$, $\mu(\{B : A \leq_{\mathrm{T}} B\}) = 0$.

COMPLEXITY THEORY:

THEOREM (A-S, 1986). For any set $A \notin \mathrm{P}$, $\mu(\{B : A \leq_{\mathrm{m}}^{\mathrm{P}} B\}) = 0$.

By countable additivity of μ this implies:

COROLLARY. $\mu(\{A : \mathrm{P} \subset \mathrm{P}_{\mathrm{m}}(A) \cap \mathrm{REC}\}) = \mu(\{A : \mathrm{P} \subset \mathrm{P}_{\mathrm{m}}(A) \cap \mathrm{E}\}) = 0$. I.e., the class of E-useful sets has measure 0.

(The question whether the full analog of Sacks's theorem holds in complexity theory is equivalent to the $\mathrm{BPP} =? \mathrm{P}$-Problem (Bennet and Gill 1981, A-S 1986), hence one of the fundamental open problems in this area!)

What weak hardness notions are typical among all sets ?

What weak hardness notions are typical among all sets ?

By the last corollary we get:

	E	EXP	EL	REC	ALL
hard	?	?	?	?	0
measure hard	?	?	?	?	0
strongly non-trivial	?	?	?	?	0
nontrivial	?	?	?	?	0
useful	1	1	1	?	0

What weak hardness notions are typical among all sets ?

By the last corollary we get:

	E	EXP	EL	REC	ALL
hard	?	?	?	?	0
measure hard	?	?	?	?	0
strongly non-trivial	?	?	?	?	0
nontrivial	?	?	?	?	0
useful	1	1	1	?	0

The corollary can be effectivized as follows:

THEOREM. Let A be *rec*-random. Then A is E-useless. In fact, $\mathrm{P_m}(A) \cap \mathrm{REC} = \mathrm{P}$.

What weak hardness notions are typical among all sets ?

By the last corollary we get:

	E	EXP	EL	REC	ALL
hard	?	?	?	?	0
measure hard	?	?	?	?	0
strongly non-trivial	?	?	?	?	0
nontrivial	?	?	?	?	0
useful	1	1	1	?	0

The corollary can be effectivized as follows:

THEOREM. Let A be *rec*-random. Then A is E-useless. In fact, $\mathrm{P_m}(A) \cap \mathrm{REC} = \mathrm{P}$.

PROOF (IDEA). If a computable set $B \notin \mathrm{P}$ is P-m-reducible to a set A via f then $f(B)$ is an infinite c.e. subset of A, hence A is not REC-immune (whereas *rec*-random sets are REC-(bi-)immune).

$\mathbb{E}$-Hard sets are untypical everywhere!

$\mathbb{E}$-Hard sets are untypical everywhere!

Bi-immunity properties of pseudo-random sets can be used to show that $\mathbb{E}$-hardness is untypical in all of the settings we consider here:

E-Hard sets are untypical everywhere!

Bi-immunity properties of pseudo-random sets can be used to show that E-hardness is untypical in all of the settings we consider here:

- THEOREM 1 (Berman, 1976) No E-hard set is P-immune.

E-Hard sets are untypical everywhere!

Bi-immunity properties of pseudo-random sets can be used to show that E-hardness is untypical in all of the settings we consider here:

- THEOREM 1 (Berman, 1976) No E-hard set is P-immune.
- THEOREM 2 (Mayordomo, 1994) The class of P-biimmune sets has p-measure 1 (hence measure 1 in E, EXP, EL and REC, and classical measure 1).

E-Hard sets are untypical everywhere!

Bi-immunity properties of pseudo-random sets can be used to show that E-hardness is untypical in all of the settings we consider here:

- THEOREM 1 (Berman, 1976) No E-hard set is P-immune.
- THEOREM 2 (Mayordomo, 1994) The class of P-biimmune sets has p-measure 1 (hence measure 1 in E, EXP, EL and REC, and classical measure 1).
- COROLLARY (Mayordomo, 1994) The class of E-hard sets has measure 0 in E, EXP, EL, REC and ALL.

E-Hard sets are untypical everywhere!

Bi-immunity properties of pseudo-random sets can be used to show that E-hardness is untypical in all of the settings we consider here:

- THEOREM 1 (Berman, 1976) No E-hard set is P-immune.
- THEOREM 2 (Mayordomo, 1994) The class of P-biimmune sets has *p*-measure 1 (hence measure 1 in E, EXP, EL and REC, and classical measure 1).
- COROLLARY (Mayordomo, 1994) The class of E-hard sets has measure 0 in E, EXP, EL, REC and ALL.

Mayordomo's Theorem 2 and Corollary has been rephrased in terms of randomness as follows.

THEOREM (A-S, Terwijn, Zheng, 1997). Any *n*-random set is P-biimmune hence not E-hard.

By Mayordomo's results:

	E	EXP	EL	REC	ALL
hard	0	0	0	0	0
measure hard	?	?	?	?	0
strongly non-trivial	?	?	?	?	0
nontrivial	?	?	?	?	0
useful	1	1	1	?	0

We next look at the question of typicalness of the weak hardness notions in E (i.e., typicalness of the weak completeness notions).

What weak hardness notions are typical in E ?

What weak hardness notions are typical in E ?

THEOREM (A-S, Terwijn, Zheng, 1997). Any n^2-random set $A \in \mathrm{E}$ is E-measure-hard.

What weak hardness notions are typical in E ?

THEOREM (A-S, Terwijn, Zheng, 1997). Any n^2-random set $A \in \mathrm{E}$ is E-measure-hard.

Since, by the randomness characterization of the measure in E, a set A is E-measure-hard if and only if for every $k \geq 1$ there is an n^k-random set $A_k \in \mathrm{E}$ such that $A_k \leq_{\mathrm{m}}^{\mathrm{P}} A$, this is immediate by the following lemma.

What weak hardness notions are typical in E ?

THEOREM (A-S, Terwijn, Zheng, 1997). Any n^2-random set $A \in \mathrm{E}$ is E-measure-hard.

Since, by the randomness characterization of the measure in E, a set A is E-measure-hard if and only if for every $k \geq 1$ there is an n^k-random set $A_k \in \mathrm{E}$ such that $A_k \leq^{\mathrm{P}}_{\mathrm{m}} A$, this is immediate by the following lemma.

RANDOMNESS EXPANSION LEMMA (A-S, Terwijn, Zheng, 1997). Let $A \in E$ be n^2-random. Then, for $k \geq 1$, there are sets A_k and A'_k such that

- A_k is n^k-random, $A_k \leq^{\mathrm{P}}_{\mathrm{m}} A$, and $A_k \in \mathrm{E}$
- A'_k is $2^{(log\ n)^k}$-random, $A'_k \leq^{\mathrm{P}}_{\mathrm{m}} A$ (and $A'_k \in \mathrm{EXP}$)

What weak hardness notions are typical in E ?

THEOREM (A-S, Terwijn, Zheng, 1997). Any n^2-random set $A \in \mathrm{E}$ is E-measure-hard.

Since, by the randomness characterization of the measure in E, a set A is E-measure-hard if and only if for every $k \geq 1$ there is an n^k-random set $A_k \in \mathrm{E}$ such that $A_k \leq_{\mathrm{m}}^{\mathrm{P}} A$, this is immediate by the following lemma.

RANDOMNESS EXPANSION LEMMA (A-S, Terwijn, Zheng, 1997). Let $A \in E$ be n^2-random. Then, for $k \geq 1$, there are sets A_k and A'_k such that

- A_k is n^k-random, $A_k \leq_{\mathrm{m}}^{\mathrm{P}} A$, and $A_k \in \mathrm{E}$
- A'_k is $2^{(log\ n)^k}$-random, $A'_k \leq_{\mathrm{m}}^{\mathrm{P}} A$ (and $A'_k \in \mathrm{EXP}$)

PROOF (IDEA). Let $A_k = \{x : 0^{k \cdot |x|} x \in A\}$ and $A'_k = \{x : 0^{|x|^{k+1}} x \in A\}$.

What weak hardness notions are typical in E ?

What weak hardness notions are typical in E ?

What weak hardness notions are typical in E ?

Since the class of n^2-random sets has p-measure 1, the class of n^2-random sets in E has measure 1 in E. So the theorem of A-S, Terwijn and Zheng implies:

What weak hardness notions are typical in E ?

Since the class of n^2-random sets has p-measure 1, the class of n^2-random sets in E has measure 1 in E. So the theorem of A-S, Terwijn and Zheng implies:

COROLLARY (A-S, Terwijn, Zheng, 1997). The class of E-measure-hard sets has measure 1 in E.

What weak hardness notions are typical in E ?

Since the class of n^2-random sets has p-measure 1, the class of n^2-random sets in E has measure 1 in E. So the theorem of A-S, Terwijn and Zheng implies:

COROLLARY (A-S, Terwijn, Zheng, 1997). The class of E-measure-hard sets has measure 1 in E.

	E	EXP	EL	REC	ALL
hard	0	0	0	0	0
measure hard	1	?	?	?	0
strongly non-trivial	1	?	?	?	0
nontrivial	1	?	?	?	0
useful	1	1	1	?	0

What weak hardness notions are typical in EXP ?

What weak hardness notions are typical in EXP ?

THEOREM (Juedes and Lutz, 1995). Let A be p-random. Then A is not E-measure hard.

What weak hardness notions are typical in EXP ?

THEOREM (Juedes and Lutz, 1995). Let A be p-random. Then A is not E-measure hard.

PROOF (IDEA). It suffices to show that there is no n^2-random set $B \in \mathrm{E}$ such that $B \leq_{\mathrm{m}}^{\mathrm{P}} A$.

What weak hardness notions are typical in EXP ?

THEOREM (Juedes and Lutz, 1995). Let A be p-random. Then A is not E-measure hard.

PROOF (IDEA). It suffices to show that there is no n^2-random set $B \in \mathrm{E}$ such that $B \leq_{\mathrm{m}}^{\mathrm{P}} A$. For a contradiction assume that $B \leq_{\mathrm{m}}^{\mathrm{P}} A$ via f where $B \in E_k$ is n^2-random.

What weak hardness notions are typical in EXP ?

THEOREM (Juedes and Lutz, 1995). Let A be p-random. Then A is not E-measure hard.

PROOF (IDEA). It suffices to show that there is no n^2-random set $B \in \mathrm{E}$ such that $B \leq^{\mathrm{P}}_{\mathrm{m}} A$. For a contradiction assume that $B \leq^{\mathrm{P}}_{\mathrm{m}} A$ via f where $B \in E_k$ is n^2-random. Then, by n^2-randomness of B, f cannot compress B, i.e., $|f(x)| \geq |x|$ for infinitely many x.

What weak hardness notions are typical in EXP ?

THEOREM (Juedes and Lutz, 1995). Let A be p-random. Then A is not E-measure hard.

PROOF (IDEA). It suffices to show that there is no n^2-random set $B \in \mathrm{E}$ such that $B \leq_{\mathrm{m}}^{\mathrm{P}} A$. For a contradiction assume that $B \leq_{\mathrm{m}}^{\mathrm{P}} A$ via f where $B \in E_k$ is n^2-random. Then, by n^2-randomness of B, f cannot compress B, i.e., $|f(x)| \geq |x|$ for infinitely many x. Since $D = \{f(x) : |f(x)| \geq |x|\} \in \mathrm{E}_2$, it follows with $B \in E_k$ that A is not $\mathrm{E}_{\max(k,2)}$-biimmune, hence not n^{k+2}-random.

What weak hardness notions are typical in EXP ?

THEOREM (Juedes and Lutz, 1995). Let A be p-random. Then A is not E-measure hard.

PROOF (IDEA). It suffices to show that there is no n^2-random set $B \in \mathrm{E}$ such that $B \leq^{\mathrm{P}}_{\mathrm{m}} A$. For a contradiction assume that $B \leq^{\mathrm{P}}_{\mathrm{m}} A$ via f where $B \in E_k$ is n^2-random. Then, by n^2-randomness of B, f cannot compress B, i.e., $|f(x)| \geq |x|$ for infinitely many x. Since $D = \{f(x) : |f(x)| \geq |x|\} \in \mathrm{E}_2$, it follows with $B \in E_k$ that A is not $\mathrm{E}_{\max(k,2)}$-biimmune, hence not n^{k+2}-random.

Since the class of p-random sets has measure 1 in EXP, EL, REC, it follows:

What weak hardness notions are typical in EXP ?

THEOREM (Juedes and Lutz, 1995). Let A be p-random. Then A is not E-measure hard.

PROOF (IDEA). It suffices to show that there is no n^2-random set $B \in \mathrm{E}$ such that $B \leq_m^P A$. For a contradiction assume that $B \leq_m^P A$ via f where $B \in E_k$ is n^2-random. Then, by n^2-randomness of B, f cannot compress B, i.e., $|f(x)| \geq |x|$ for infinitely many x. Since $D = \{f(x) : |f(x)| \geq |x|\} \in \mathrm{E}_2$, it follows with $B \in E_k$ that A is not $\mathrm{E}_{\max(k,2)}$-biimmune, hence not n^{k+2}-random.

Since the class of p-random sets has measure 1 in EXP, EL, REC, it follows:

COROLLARY (Juedes and Lutz, 1995). The class of E-measure hard sets has measure 0 in EXP, EL and REC.

What weak hardness notions are typical in EXP ?

By Juedes and Lutz (1995):

	E	EXP	EL	REC	ALL
hard	0	0	0	0	0
measure hard	1	0	0	0	0
strongly non-trivial	1	?	?	?	0
nontrivial	1	?	?	?	0
useful	1	1	1	?	0

What weak hardness notions are typical in EXP ?

By Juedes and Lutz (1995):

	E	EXP	EL	REC	ALL
hard	0	0	0	0	0
measure hard	1	0	0	0	0
strongly non-trivial	1	?	?	?	0
nontrivial	1	?	?	?	0
useful	1	1	1	?	0

This leaves for EXP the question whether the E-strongly nontrivial sets and the E-nontrivial sets are typical or untypical in EXP. As we will show neither is the case.

What weak hardness notions are typical in EXP ?

What weak hardness notions are typical in EXP ?

We first show that the strongly E-nontrivial sets are not untypical in EXP, i.e., do not have measure 0 in EXP. By the characterization of the measure in EXP in terms of randomness, it suffices to show.

What weak hardness notions are typical in EXP ?

We first show that the strongly E-nontrivial sets are not untypical in EXP, i.e., do not have measure 0 in EXP. By the characterization of the measure in EXP in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). For any $k \geq 1$ there is a $2^{(\log n)^k}$-random set $A_k \in \mathrm{EXP}$ such that A_k is strongly E-nontrivial.

What weak hardness notions are typical in EXP ?

We first show that the strongly E-nontrivial sets are not untypical in EXP, i.e., do not have measure 0 in EXP. By the characterization of the measure in EXP in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). For any $k \geq 1$ there is a $2^{(\log n)^k}$-random set $A_k \in \mathrm{EXP}$ such that A_k is strongly E-nontrivial.

PROOF (IDEA).

What weak hardness notions are typical in EXP ?

We first show that the strongly E-nontrivial sets are not untypical in EXP, i.e., do not have measure 0 in EXP. By the characterization of the measure in EXP in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). For any $k \geq 1$ there is a $2^{(\log n)^k}$-random set $A_k \in \mathrm{EXP}$ such that A_k is strongly E-nontrivial.

PROOF (IDEA).

- Fix an n^2-random set $A \in \mathrm{E}$.

What weak hardness notions are typical in EXP ?

We first show that the strongly E-nontrivial sets are not untypical in EXP, i.e., do not have measure 0 in EXP. By the characterization of the measure in EXP in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). For any $k \geq 1$ there is a $2^{(\log n)^k}$-random set $A_k \in \mathrm{EXP}$ such that A_k is strongly E-nontrivial.

PROOF (IDEA).

- Fix an n^2-random set $A \in \mathrm{E}$.
- Then, by the Randomness Expansion Lemma, $A'_k = \{x : 0^{|x|^{k+1}} x \in A\}$ is $2^{(log\ n)^k}$-random and $A'_k \in \mathrm{EXP}$.

What weak hardness notions are typical in EXP ?

We first show that the strongly E-nontrivial sets are not untypical in EXP, i.e., do not have measure 0 in EXP. By the characterization of the measure in EXP in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). For any $k \geq 1$ there is a $2^{(\log n)^k}$-random set $A_k \in \mathrm{EXP}$ such that A_k is strongly E-nontrivial.

PROOF (IDEA).

- Fix an n^2-random set $A \in \mathrm{E}$.
- Then, by the Randomness Expansion Lemma, $A'_k = \{x : 0^{|x|^{k+1}} x \in A\}$ is $2^{(log\ n)^k}$-random and $A'_k \in \mathrm{EXP}$.
- So it suffices to show that A'_k is strongly E-nontrivial, i.e., that, for any $k' \geq 1$, there is an $\mathrm{E}_{k'}$-biimmune set $B_{k'} \in \mathrm{E}$ such that $B_{k'} \leq^{\mathrm{P}}_{\mathrm{m}} A'_k$.

What weak hardness notions are typical in EXP ?

We first show that the strongly E-nontrivial sets are not untypical in EXP, i.e., do not have measure 0 in EXP. By the characterization of the measure in EXP in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). For any $k \geq 1$ there is a $2^{(\log n)^k}$-random set $A_k \in \mathrm{EXP}$ such that A_k is strongly E-nontrivial.

PROOF (IDEA).

- Fix an n^2-random set $A \in \mathrm{E}$.
- Then, by the Randomness Expansion Lemma, $A'_k = \{x :\ 0^{|x|^{k+1}} x \in A\}$ is $2^{(log\ n)^k}$-random and $A'_k \in \mathrm{EXP}$.
- So it suffices to show that A'_k is strongly E-nontrivial, i.e., that, for any $k' \geq 1$, there is an $\mathrm{E}_{k'}$-biimmune set $B_{k'} \in \mathrm{E}$ such that $B_{k'} \leq^{\mathrm{P}}_{\mathrm{m}} A'_k$.
- In fact, by a Biimunity Expansion Lemma, it suffices to give an E_1-biimmune set $B_1 \in \mathrm{E}$ such that $B_1 \leq^{\mathrm{P}}_{\mathrm{m}} A'_k$.

What weak hardness notions are typical in EXP ?

We first show that the strongly E-nontrivial sets are not untypical in EXP, i.e., do not have measure 0 in EXP. By the characterization of the measure in EXP in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). For any $k \geq 1$ there is a $2^{(\log n)^k}$-random set $A_k \in \mathrm{EXP}$ such that A_k is strongly E-nontrivial.

PROOF (IDEA).

- Fix an n^2-random set $A \in \mathrm{E}$.
- Then, by the Randomness Expansion Lemma, $A'_k = \{x : 0^{|x|^{k+1}} x \in A\}$ is $2^{(log\ n)^k}$-random and $A'_k \in \mathrm{EXP}$.
- So it suffices to show that A'_k is strongly E-nontrivial, i.e., that, for any $k' \geq 1$, there is an $\mathrm{E}_{k'}$-biimmune set $B_{k'} \in \mathrm{E}$ such that $B_{k'} \leq^{\mathrm{P}}_{\mathrm{m}} A'_k$.
- In fact, by a Biimunity Expansion Lemma, it suffices to give an E_1-biimmune set $B_1 \in \mathrm{E}$ such that $B_1 \leq^{\mathrm{P}}_{\mathrm{m}} A'_k$.
- $B_1 = \{yx : |x|^{k+1} \leq |y| < |x+1|^{k+1}\ \&\ x \in A'_k\}$ will do.

What weak hardness notions are typical in EXP ?

What weak hardness notions are typical in EXP ?

We now show that the E-trivial sets are not untypical in EXP either, i.e., do not have measure 0 in EXP. Again, by the characterization of the measure in EXP in terms of randomness, it suffices to show.

What weak hardness notions are typical in EXP ?

We now show that the E-trivial sets are not untypical in EXP either, i.e., do not have measure 0 in EXP. Again, by the characterization of the measure in EXP in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). For any $k \geq 1$ there is a $2^{(\log n)^k}$-random set $A_k \in \mathrm{EXP}$ such that A_k is E-trivial.

What weak hardness notions are typical in EXP ?

We now show that the E-trivial sets are not untypical in EXP either, i.e., do not have measure 0 in EXP. Again, by the characterization of the measure in EXP in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). For any $k \geq 1$ there is a $2^{(\log n)^k}$-random set $A_k \in \mathrm{EXP}$ such that A_k is E-trivial.

By downward closure of E-triviality and by the Randomness Expansion Lemma, it suffices to prove:

What weak hardness notions are typical in EXP ?

We now show that the E-trivial sets are not untypical in EXP either, i.e., do not have measure 0 in EXP. Again, by the characterization of the measure in EXP in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). For any $k \geq 1$ there is a $2^{(\log n)^k}$-random set $A_k \in \mathrm{EXP}$ such that A_k is E-trivial.

By downward closure of E-triviality and by the Randomness Expansion Lemma, it suffices to prove:

MAIN LEMMA (A-S and Bakibayev, TOCS ta). There is an n^2-random set $A \in \mathrm{EXP}$ which is E-trivial.

What weak hardness notions are typical in EXP ?

We now show that the E-trivial sets are not untypical in EXP either, i.e., do not have measure 0 in EXP. Again, by the characterization of the measure in EXP in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). For any $k \geq 1$ there is a $2^{(\log n)^k}$-random set $A_k \in \mathrm{EXP}$ such that A_k is E-trivial.

By downward closure of E-triviality and by the Randomness Expansion Lemma, it suffices to prove:

MAIN LEMMA (A-S and Bakibayev, TOCS ta). There is an n^2-random set $A \in \mathrm{EXP}$ which is E-trivial.

We come back to the proof later!

What weak hardness notions are typical in EXP ?

We now show that the E-trivial sets are not untypical in EXP either, i.e., do not have measure 0 in EXP. Again, by the characterization of the measure in EXP in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). For any $k \geq 1$ there is a $2^{(\log n)^k}$-random set $A_k \in \mathrm{EXP}$ such that A_k is E-trivial.

By downward closure of E-triviality and by the Randomness Expansion Lemma, it suffices to prove:

MAIN LEMMA (A-S and Bakibayev, TOCS ta). There is an n^2-random set $A \in \mathrm{EXP}$ which is E-trivial.

We come back to the proof later! A direct proof of the Theorem can be given along the same lines (only the notation and some of the calculations are more tedious).

What weak hardness notions are typical in EXP ?

We now show that the E-trivial sets are not untypical in EXP either, i.e., do not have measure 0 in EXP. Again, by the characterization of the measure in EXP in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). For any $k \geq 1$ there is a $2^{(\log n)^k}$-random set $A_k \in \mathrm{EXP}$ such that A_k is E-trivial.

By downward closure of E-triviality and by the Randomness Expansion Lemma, it suffices to prove:

MAIN LEMMA (A-S and Bakibayev, TOCS ta). There is an n^2-random set $A \in \mathrm{EXP}$ which is E-trivial.

We come back to the proof later! A direct proof of the Theorem can be given along the same lines (only the notation and some of the calculations are more tedious). In fact, the proof can be modified, in order to build for elementary recursive (time-constructible) $t(n)$ an E-trivial $t(n)$-random set $A \in \mathrm{EL}$ ($A \in \mathrm{REC}$). So the E-trivial sets in the classes EL and REC are not untypical either.

What weak hardness notions are typical in EXP ?

What weak hardness notions are typical in EXP ?

By the preceding results we can extend the diagram as follows:

	E	EXP	EL	REC	ALL
hard	0	0	0	0	0
measure hard	1	0	0	0	0
strongly non-trivial	1	$\neq 0, \neq 1$	$(\neq 1), = 0?$	$(\neq 1), = 0?$	0
nontrivial	1	$\neq 0, \neq 1$	$(\neq 1), = 0?$	$(\neq 1), = 0?$	0
useful	1	1	1	?	0

What weak hardness notions are typical in EXP ?

By the preceding results we can extend the diagram as follows:

	E	EXP	EL	REC	ALL
hard	0	0	0	0	0
measure hard	1	0	0	0	0
strongly non-trivial	1	$\neq 0, \neq 1$	$(\neq 1), = 0?$	$(\neq 1), = 0?$	0
nontrivial	1	$\neq 0, \neq 1$	$(\neq 1), = 0?$	$(\neq 1), = 0?$	0
useful	1	1	1	?	0

For EL and REC we have argued already that the nontrivial sets (hence the strongly nontrivial sets) are not typical. This leaves the question whether they are untypical or neither typical nor untypical.

Strongly nontrivial sets are untypical in EL and REC

Strongly nontrivial sets are untypical in EL and REC

THEOREM (A-S and Bakibayev). The class of strongly E-nontrivial sets has measure 0 in EL and REC.

Strongly nontrivial sets are untypical in EL and REC

THEOREM (A-S and Bakibayev). The class of strongly E-nontrivial sets has measure 0 in EL and REC.

By the characterization of the measure in these classes in terms of randomness, it suffices to show.

Strongly nontrivial sets are untypical in EL and REC

THEOREM (A-S and Bakibayev). The class of strongly E-nontrivial sets has measure 0 in EL and REC.

By the characterization of the measure in these classes in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). Let A be $exp_5(n)$-random (where $exp_5(n)$ is the 5-times iterated exponential function). Then A is not strongly E-nontrivial.

Strongly nontrivial sets are untypical in EL and REC

THEOREM (A-S and Bakibayev). The class of strongly E-nontrivial sets has measure 0 in EL and REC.

By the characterization of the measure in these classes in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). Let A be $exp_5(n)$-random (where $exp_5(n)$ is the 5-times iterated exponential function). Then A is not strongly E-nontrivial.

This is immediate by the following lemma, since $exp_5(n)$-random sets are $\mathrm{DTIME}(exp_5(n))$-biimmune.

LEMMA (A-S and Bakibayev). Let A and B be sets such that A is $\mathrm{DTIME}(exp_5(n))$-biimmune, $B \in \mathrm{E}$ and $B \leq^{\mathrm{P}}_{\mathrm{m}} A$. Then B is not E_2-biimmune.

Strongly nontrivial sets are untypical in EL and REC

THEOREM (A-S and Bakibayev). The class of strongly E-nontrivial sets has measure 0 in EL and REC.

By the characterization of the measure in these classes in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). Let A be $exp_5(n)$-random (where $exp_5(n)$ is the 5-times iterated exponential function). Then A is not strongly E-nontrivial.

This is immediate by the following lemma, since $exp_5(n)$-random sets are $\mathrm{DTIME}(exp_5(n))$-biimmune.

LEMMA (A-S and Bakibayev). Let A and B be sets such that A is $\mathrm{DTIME}(exp_5(n))$-biimmune, $B \in \mathrm{E}$ and $B \leq^{\mathrm{P}}_{\mathrm{m}} A$. Then B is not E_2-biimmune.

PROOF (IDEA). By $\mathrm{DTIME}(exp_5(n))$-biimmunity a reduction f must be very strongly compressing.

Strongly nontrivial sets are untypical in EL and REC

THEOREM (A-S and Bakibayev). The class of strongly E-nontrivial sets has measure 0 in EL and REC.

By the characterization of the measure in these classes in terms of randomness, it suffices to show.

THEOREM (A-S and Bakibayev). Let A be $exp_5(n)$-random (where $exp_5(n)$ is the 5-times iterated exponential function). Then A is not strongly E-nontrivial.

This is immediate by the following lemma, since $exp_5(n)$-random sets are $\mathrm{DTIME}(exp_5(n))$-biimmune.

LEMMA (A-S and Bakibayev). Let A and B be sets such that A is $\mathrm{DTIME}(exp_5(n))$-biimmune, $B \in \mathrm{E}$ and $B \leq^{\mathrm{P}}_{\mathrm{m}} A$. Then B is not E_2-biimmune.

PROOF (IDEA). By $\mathrm{DTIME}(exp_5(n))$-biimmunity a reduction f must be very strongly compressing. So, for infinitely many x there are much shorter x' with $f(x) = f(x')$ (hence $B(x) = B(x')$) which makes the computation of $B(x)$ easy on such strings x.

Nontrivial sets are not untypical in EL and REC

Nontrivial sets are not untypical in EL and REC

We show this for REC. The proof for EL is similar. It suffices to show:

THEOREM (A-S and Bakibayev). For any strictly increasing time-constructible function $t(n)$ there is a computable $t(n)$-random set A_t which is E-nontrivial.

Nontrivial sets are not untypical in EL and REC

We show this for REC. The proof for EL is similar. It suffices to show:

THEOREM (A-S and Bakibayev). For any strictly increasing time-constructible function $t(n)$ there is a computable $t(n)$-random set A_t which is E-nontrivial.

PROOF (IDEA).

Nontrivial sets are not untypical in EL and REC

We show this for REC. The proof for EL is similar. It suffices to show:

THEOREM (A-S and Bakibayev). For any strictly increasing time-constructible function $t(n)$ there is a computable $t(n)$-random set A_t which is E-nontrivial.

PROOF (IDEA).

- Let $A \in \mathrm{E}$ be n^2-random and let $A_t = \{z_n : 0^{t(n)} \in A\}$.

Nontrivial sets are not untypical in EL and REC

We show this for REC. The proof for EL is similar. It suffices to show:

THEOREM (A-S and Bakibayev). For any strictly increasing time-constructible function $t(n)$ there is a computable $t(n)$-random set A_t which is E-nontrivial.

PROOF (IDEA).

- Let $A \in \mathrm{E}$ be n^2-random and let $A_t = \{z_n : 0^{t(n)} \in A\}$.
- Then A_t is computable and $t(n)$-random (by a Random Expansion argument).

Nontrivial sets are not untypical in EL and REC

We show this for REC. The proof for EL is similar. It suffices to show:

THEOREM (A-S and Bakibayev). For any strictly increasing time-constructible function $t(n)$ there is a computable $t(n)$-random set A_t which is E-nontrivial.

PROOF (IDEA).

- Let $A \in \mathrm{E}$ be n^2-random and let $A_t = \{z_n : 0^{t(n)} \in A\}$.
- Then A_t is computable and $t(n)$-random (by a Random Expansion argument).
- Moreover, for $D = \{0^{t(n)} : n \geq 0\}$, $D \in \mathrm{P}$ and $A \cap D \leq^{\mathrm{P}}_{\mathrm{m}} A_t$ (both by choice of t) and $A \cap D \in \mathrm{E} \setminus \mathrm{E}_1$ (by n^2-randomness of A).

Nontrivial sets are not untypical in EL and REC

We show this for REC. The proof for EL is similar. It suffices to show:

THEOREM (A-S and Bakibayev). For any strictly increasing time-constructible function $t(n)$ there is a computable $t(n)$-random set A_t which is E-nontrivial.

PROOF (IDEA).

- Let $A \in \mathrm{E}$ be n^2-random and let $A_t = \{z_n : 0^{t(n)} \in A\}$.
- Then A_t is computable and $t(n)$-random (by a Random Expansion argument).
- Moreover, for $D = \{0^{t(n)} : n \geq 0\}$, $D \in \mathrm{P}$ and $A \cap D \leq^{\mathrm{P}}_{\mathrm{m}} A_t$ (both by choice of t) and $A \cap D \in \mathrm{E} \setminus \mathrm{E}_1$ (by n^2-randomness of A).
- The claim follows since any tally set in $\mathrm{E} \setminus \mathrm{E}_1$ is E-nontrivial (and since E-nontriviality is inherited upwards).

What weak hardness notions are typical in EL and REC? - Conclusion

By the preceding results we can almost complete the diagram:

	E	EXP	EL	REC	ALL
hard	0	0	0	0	0
measure hard	1	0	0	0	0
strongly non-trivial	1	$\neq 0, \neq 1$	0	0	0
nontrivial	1	$\neq 0, \neq 1$	$\neq 0, \neq 1$	$\neq 0, \neq 1$	0
useful	1	1	1	?	0

What weak hardness notions are typical in EL and REC? - Conclusion

By the preceding results we can almost complete the diagram:

	E	EXP	EL	REC	ALL
hard	0	0	0	0	0
measure hard	1	0	0	0	0
strongly non-trivial	1	$\neq 0, \neq 1$	0	0	0
nontrivial	1	$\neq 0, \neq 1$	$\neq 0, \neq 1$	$\neq 0, \neq 1$	0
useful	1	1	1	?	0

It only remains to classify the E-useful sets in REC. Since E-nontrivial sets are E-useful, we already know that the E-useful sets are not untypical in REC. But are they typical? The answer is NO!

E-useful sets are not typical in REC

It suffices to show:

THEOREM (A-S and Bakibayev). Let B be E-complete. For any strictly increasing time-constructible function $t(n)$ there is a computable $t(n)$-random set A such that A and B are a minimal pair (i.e., A is E-useless).

E-useful sets are not typical in REC

It suffices to show:

THEOREM (A-S and Bakibayev). Let B be E-complete. For any strictly increasing time-constructible function $t(n)$ there is a computable $t(n)$-random set A such that A and B are a minimal pair (i.e., A is E-useless).

	E	EXP	EL	REC	ALL
hard	0	0	0	0	0
measure hard	1	0	0	0	0
strongly non-trivial	1	$\neq 0, \neq 1$	0	0	0
nontrivial	1	$\neq 0, \neq 1$	$\neq 0, \neq 1$	$\neq 0, \neq 1$	0
useful	1	1	1	$\neq 0, \neq 1$	0

E-useful sets are not typical in REC

It suffices to show:

THEOREM (A-S and Bakibayev). Let B be E-complete. For any strictly increasing time-constructible function $t(n)$ there is a computable $t(n)$-random set A such that A and B are a minimal pair (i.e., A is E-useless).

	E	EXP	EL	REC	ALL
hard	0	0	0	0	0
measure hard	1	0	0	0	0
strongly non-trivial	1	$\neq 0, \neq 1$	0	0	0
nontrivial	1	$\neq 0, \neq 1$	$\neq 0, \neq 1$	$\neq 0, \neq 1$	0
useful	1	1	1	$\neq 0, \neq 1$	0

The proof of the theorem combines the minimal pair technique of Ambos-Spies (1987) (which uses a speed-up-argument) with the construction of a pseudo-random set in the style of Ambos-Spies and Kräling (2009). The technical features are similar to the proof of the MAIN LEMMA. So we sketch the proof of the latter.

Proof of the Main Lemma

Proof of the Main Lemma

MAIN LEMMA. There is an n^2-random set $A \in \mathrm{EXP}$ which is E-trivial.

Proof of the Main Lemma

MAIN LEMMA. There is an n^2-random set $A \in \mathrm{EXP}$ which is E-trivial.

- We construct a set A with the required properties by an effective "slow diagonalization" or "wait-and-see argument" where $A(z_s)$ is defined at stage $s+1$ of the construction.

Proof of the Main Lemma

MAIN LEMMA. There is an n^2-random set $A \in \mathrm{EXP}$ which is E-trivial.

- We construct a set A with the required properties by an effective "slow diagonalization" or "wait-and-see argument" where $A(z_s)$ is defined at stage $s+1$ of the construction.
- Complexity. In order to guarantee $A \in \mathrm{EXP}$ we ensure that $A \in \mathrm{EXP}_2 = \mathrm{DTIME}(2^{n^2})$.

Proof of the Main Lemma

MAIN LEMMA. There is an n^2-random set $A \in \mathrm{EXP}$ which is E-trivial.

- We construct a set A with the required properties by an effective "slow diagonalization" or "wait-and-see argument" where $A(z_s)$ is defined at stage $s+1$ of the construction.
- Complexity. In order to guarantee $A \in \mathrm{EXP}$ we ensure that $A \in \mathrm{EXP}_2 = \mathrm{DTIME}(2^{n^2})$.
- n^2-randomness. In order to make A n^2-random we have to ensure that no n^2-martingale succeeds on A.

Proof of the Main Lemma

MAIN LEMMA. There is an n^2-random set $A \in \mathrm{EXP}$ which is E-trivial.

- We construct a set A with the required properties by an effective "slow diagonalization" or "wait-and-see argument" where $A(z_s)$ is defined at stage $s+1$ of the construction.
- Complexity. In order to guarantee $A \in \mathrm{EXP}$ we ensure that $A \in \mathrm{EXP}_2 = \mathrm{DTIME}(2^{n^2})$.
- n^2-randomness. In order to make A n^2-random we have to ensure that no n^2-martingale succeeds on A.

  Since there is an (normed) n^6-martingale d which is universal for the n^2-martingales, it suffices to ensure

$$(*) \ \ \forall s \, (d(A \restriction z_s) \leq 1)$$

Proof of the Main Lemma

MAIN LEMMA. There is an n^2-random set $A \in \mathrm{EXP}$ which is E-trivial.

- We construct a set A with the required properties by an effective "slow diagonalization" or "wait-and-see argument" where $A(z_s)$ is defined at stage $s+1$ of the construction.
- Complexity. In order to guarantee $A \in \mathrm{EXP}$ we ensure that $A \in \mathrm{EXP}_2 = \mathrm{DTIME}(2^{n^2})$.
- n^2-randomness. In order to make A n^2-random we have to ensure that no n^2-martingale succeeds on A.

  Since there is an (normed) n^6-martingale d which is universal for the n^2-martingales, it suffices to ensure

  $$(*)\ \ \forall\, s\, (d(A \upharpoonright z_s) \leq 1)$$

  We say we use the safe randomness strategy at stage $s+1$ if we let $A(z_s) = i$ for the least $i \leq 1$ such that $d(A \upharpoonright z_s)i \leq d(A \upharpoonright z_s)$ (such an i exists by the fairness condition).

Proof of the Main Lemma

Proof of the Main Lemma

- E-triviality.

Proof of the Main Lemma

- E-triviality. It suffices to ensure

$$(**)\ \forall B \in \mathrm{E}\,(B \leq^p_m A \Rightarrow B \in DTIME(2^n))$$

Proof of the Main Lemma

- E-triviality. It suffices to ensure

$$(**)\ \forall B \in \mathrm{E}\,(B \leq^p_m A \Rightarrow B \in DTIME(2^n))$$

This is done by some speed-up-argument.

Proof of the Main Lemma

- E-triviality. It suffices to ensure

$$(**)\ \forall B \in \mathrm{E}\ (B \leq_m^p A \Rightarrow B \in DTIME(2^n))$$

This is done by some speed-up-argument. Given (sufficiently fast) computable enumerations $\{E_e : e \geq 0\}$ and $\{f_e : e \geq 0\}$ of the sets in E (where $E_e \in \mathrm{E}_e$ uniformly) and the polynomial time reduction functions (where $f_e \in \mathrm{E}_1$), respectively, it suffices to meet the requirements

$$\Re_e : E_{e_0} \leq_m^p A \textit{ via } f_{e_1} \Rightarrow \forall^\infty x \left(|x| > \frac{1}{2^e} \cdot |f_{e_1}(x)|^2\right)$$

($e = \langle e_0, e_1 \rangle$) and to ensure

$$(***)\ \ \forall e > 0\ (A \in \mathrm{DTIME}(2^{\frac{1}{2^e} \cdot n^2})).$$

Proof of the Main Lemma

- E-triviality. It suffices to ensure

$$(**)\ \forall B \in \mathrm{E}\, (B \leq^p_m A \Rightarrow B \in DTIME(2^n))$$

This is done by some speed-up-argument. Given (sufficiently fast) computable enumerations $\{E_e : e \geq 0\}$ and $\{f_e : e \geq 0\}$ of the sets in E (where $E_e \in \mathrm{E}_e$ uniformly) and the polynomial time reduction functions (where $f_e \in \mathrm{E}_1$), respectively, it suffices to meet the requirements

$$\Re_e : E_{e_0} \leq^p_m A \text{ via } f_{e_1} \Rightarrow \forall^\infty x\, (|x| > \frac{1}{2^e} \cdot |f_{e_1}(x)|^2)$$

$(e = \langle e_0, e_1 \rangle)$ and to ensure

$$(***)\ \forall e > 0\, (A \in \mathrm{DTIME}(2^{\frac{1}{2^e} \cdot n^2})).$$

If f_{e_1} is not sufficiently compressing then $\Re_e$ is met by diagonalization. (Time required for doing so: about $2^{\frac{1}{e} \cdot n^2}$, i.e., decreasing for growing e. Since the requirements are finitary, this is consistent with $(***)$!)

Proof of the Main Lemma

Proof of the Main Lemma

- Potential conflicts.

Proof of the Main Lemma

- Potential conflicts. Meeting a requirement $\Re_e$ by appropriately setting $A(z_s) = j$ (some j) may not be compatible with the safe randomness strategy and may result in an increase of d yielding $d([A \upharpoonright z_s]j) > 1$ which is not compatible with $(*)$.

Proof of the Main Lemma

- Potential conflicts. Meeting a requirement $\mathfrak{R}_e$ by appropriately setting $A(z_s) = j$ (some j) may not be compatible with the safe randomness strategy and may result in an increase of d yielding $d([A \upharpoonright z_s]j) > 1$ which is not compatible with $(*)$.

  This conflict is resolved as follows.

Proof of the Main Lemma

- Potential conflicts. Meeting a requirement $\Re_e$ by appropriately setting $A(z_s) = j$ (some j) may not be compatible with the safe randomness strategy and may result in an increase of d yielding $d([A \restriction z_s]j) > 1$ which is not compatible with $(*)$.

  This conflict is resolved as follows.

  - Each requirement $\Re_e$ is provided with an account where the original balance b_e is 0.

Proof of the Main Lemma

- Potential conflicts. Meeting a requirement $\mathfrak{R}_e$ by appropriately setting $A(z_s) = j$ (some j) may not be compatible with the safe randomness strategy and may result in an increase of d yielding $d([A \upharpoonright z_s]j) > 1$ which is not compatible with $(*)$.

  This conflict is resolved as follows.

  - Each requirement $\mathfrak{R}_e$ is provided with an account where the original balance b_e is 0.

    So, at the beginning, $d(A \upharpoonright z_0) + \sum_{e=0}^{\infty} b_e(0) = 1 + 0 \leq 1$. And this bound will be preserved throughout the construction.

    (So $(*)$ is satisfied.)

Proof of the Main Lemma

Proof of the Main Lemma

- Now, if $\mathfrak{R}_e$ is elected to act at stage $s+1$ and the current balance $b_e(s)$ suffices to compensate the increase in d, i.e., if

  $$d([A \upharpoonright z_s]j) \leq d(A \upharpoonright z_s) + b_e(s)$$

  then $\mathfrak{R}_e$ is allowed to act and its account is set to 0 ($b_e(s+1) = 0$).

Proof of the Main Lemma

- Now, if $\Re_e$ is elected to act at stage $s+1$ and the current balance $b_e(s)$ suffices to compensate the increase in d, i.e., if

$$d([A \upharpoonright z_s]j) \leq d(A \upharpoonright z_s) + b_e(s)$$

then $\Re_e$ is allowed to act and its account is set to 0 ($b_e(s+1) = 0$).

Otherwise the safe randomness strategy is performed and the resulting decrease in d, i.e.,

$$d_{s+1} := d(A \upharpoonright z_s) - d([A \upharpoonright z_s]A(z_s))$$

is payed into the account of $\Re_e$ ($b_e(s+1) = b_e(s) + d_{s+1}$).

Proof of the Main Lemma

- Now, if $\Re_e$ is elected to act at stage $s+1$ and the current balance $b_e(s)$ suffices to compensate the increase in d, i.e., if

$$d([A \upharpoonright z_s]j) \leq d(A \upharpoonright z_s) + b_e(s)$$

then $\Re_e$ is allowed to act and its account is set to 0 ($b_e(s+1) = 0$).

Otherwise the safe randomness strategy is performed and the resulting decrease in d, i.e.,

$$d_{s+1} := d(A \upharpoonright z_s) - d([A \upharpoonright z_s]A(z_s))$$

is payed into the account of $\Re_e$ ($b_e(s+1) = b_e(s) + d_{s+1}$).

Note that in this case $b_e(s+1) > 2 \cdot b_e(s)$ by the fairness condition. So, eventually, $\Re_e$ has enough money in its account in order to pay for the desired diagonalization step. (Again, by the fairness condition, $\Re_e$ has never to pay more than 1 while if $\Re_e$ would be prevented from acting infinitely often, the balance $b_e(s)$ would go to infinity.)

Conclusion

Conclusion

The measure results we have obtained yield the following results on randomness telling us how much randomness is compatible with what knowledge of E:

Conclusion

The measure results we have obtained yield the following results on randomness telling us how much randomness is compatible with what knowledge of E:

(i) No n-random set is E-hard.

Conclusion

The measure results we have obtained yield the following results on randomness telling us how much randomness is compatible with what knowledge of E:

(i) No n-random set is E-hard.

(ii) No p-random set is E-measure hard (whereas there are n^k-random sets which are E-measure hard).

Conclusion

The measure results we have obtained yield the following results on randomness telling us how much randomness is compatible with what knowledge of E:

(i) No n-random set is E-hard.

(ii) No p-random set is E-measure hard (whereas there are n^k-random sets which are E-measure hard).

(iii) No $\exp_5(n)$-random set is strongly E-nontrivial.

Conclusion

The measure results we have obtained yield the following results on randomness telling us how much randomness is compatible with what knowledge of E:

(i) No *n*-random set is E-hard.

(ii) No *p*-random set is E-measure hard (whereas there are n^k-random sets which are E-measure hard).

(iii) No $\exp_5(n)$-random set is strongly E-nontrivial.

(iv) No *rec*-random set is E-useful or even E-nontrivial (whereas, for each computable t there are $t(n)$-random sets which are E-nontrivial hence E-useful).

Conclusion

The measure results we have obtained yield the following results on randomness telling us how much randomness is compatible with what knowledge of E:

(i) No *n*-random set is E-hard.

(ii) No *p*-random set is E-measure hard (whereas there are n^k-random sets which are E-measure hard).

(iii) No $\exp_5(n)$-random set is strongly E-nontrivial.

(iv) No *rec*-random set is E-useful or even E-nontrivial (whereas, for each computable t there are $t(n)$-random sets which are E-nontrivial hence E-useful).

So (i), (ii) and (iv) are optimal whereas (iii) is not optimal (due to very generous estimates of the required upper bounds). We do not not know the optimal bound here.

THANK YOU !