Are Typical Sets Weakly Hard For Exponential Time?

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Abstract. We discuss the question whether a typical set in E = DTIME(2^{lin}), a typical set in EXP = DTIME(2^{poly}), a typical computable set, or a typical set in general is weakly hard for E in the sense of Lutz [14] (i.e., measure hard), in the sense of Ambos-Spies [3] (i.e., category hard) or in the sense of Ambos-Spies and Bakibayev [4] (i.e., nontrivial or strongly nontrivial). We will show that the answer depends on both, the base class we choose and the weak hardness notion we consider.

1 Introduction

The standard way for proving a problem to be intractable is to show that the problem is hard for the linear-exponential-time class E = DTIME(2^{lin}) under polynomial-time-bounded many-one reducibility (*p-m*-reducibility for short). Lutz [14] proposed a generalization of this approach by relaxing hardness as follows. While a set A is hard for E if *all* problems in E can be reduced to A, Lutz proposed to call a set A weakly hard if a *nonnegligible* part of E can be reduced to A. He formalized this idea by introducing a resource-bounded measure on E and by saying that a subclass of E is negligible if it has measure 0 in E. In the following this approach has been further generalized. So Ambos-Spies [3] introduced a weak hardness based on Baire category in place of measure and, more recently, Ambos-Spies and Bakibayev [4] introduced some further, much less technical generalizations of weak hardness, called strong nontriviality and nontriviality, where the latter may be viewed as the weakest weak hardness notion for E.

Though Lutz [14] has shown that there are measure hard sets (i.e., weakly hard sets in his sense) which are not hard, Ambos-Spies [3] has shown that there are category hard sets which are not measure hard, and Ambos-Spies and Bakibayev [4] have shown that there are nontrivial sets which are not strongly nontrivial and strongly nontrivial sets which are not category hard, one may ask whether these separations - or what of these separations - apply to *typical* sets.

If we look at all (not necessarily computable) sets then the weakly hard sets (in any of the above senses) are as rare as the hard sets, namely the class of nontrivial sets has (Lebesgue) measure 0. This is an easy consequence of some more general result in [1] (see Section 3 below). If we consider only sets in E, i.e., compare completeness with weak completeness, however, then the situtation becomes different. Namely, Mayordomo [16] has shown that the hard sets have measure 0 in E (in the sense of Lutz's resource bounded measure theory) whereas Ambos-Spies et al. [8] have shown that the class of measure hard sets has measure 1 in E. So a typical set in E is weakly hard but not hard.

Here we analyze typicalness of the weak hardness notions for some intermediate classes between E and the class of all sets, namely for the polynomial-exponential-time class EXP and for the class REC of computable sets. We show that among the sets in EXP not only the E-hard sets but also the measure and category hard sets for E are rare (i.e., have measure 0 in EXP) whereas the nontrivial and strongly nontrivial sets are not rare (i.e., do not have measure 0 in EXP) but also not typical (i.e., do not have measure 1 in EXP). Finally, for the class of the computable sets, only the nontrivial sets are not rare (i.e., do not have computable measure 0) wereas the sets with any of the stronger weak hardness properties for E are rare.

The outline of the paper is as follows. After introducing the relevant concepts and some basic facts on them in Section 2, in Sections 3 - 6 we shortly review the results which can be found (explicitly or implicitly) in the literature before in Sections 7 - 10 we present our new results. Due to lack of space in most cases only some of the ideas underlying the proofs are given.

Our notation is standard (see e.g. the monographs of Balcázar et al. [10] and [11]). The exponential time classes we will deal with are the classes

$$\mathrm{E} = \bigcup_{k\geq 1} \mathrm{DTIME}(2^{kn}) \quad \text{and} \quad \mathrm{EXP} = \bigcup_{k\geq 1} \mathrm{DTIME}(2^{n^k})$$

where we abbreviate the individual levels of these classes by

$$\mathrm{E}_k = \mathrm{DTIME}(2^{kn}) \quad \text{and} \quad \mathrm{EXP}_k = \mathrm{DTIME}(2^{n^k}).$$

The class of computable (recursive) problems is denoted by REC. For comparing problems we use the polynomial-time-bounded version of many-one reducibility (p-m-reducibility for short) where a set A is p-m-reducible to a set B ($A \leq^p_m B$) via f if f is polynomial-time computable and $A(x) = B(f(x))$ for all strings x. We let $\mathrm{P}_{\leq}(A) = \{B : B \leq^p_m A\}$ and $\mathrm{P}_{\geq}(A) = \{B : A \leq^p_m B\}$ denote the class of predecessors respectively successors of A under p-m-reducibility.

2 Weak hardness: basic definitions and facts

Following Lutz [14] we call a set A weakly hard for E if a nonneglibile part of E can be p-m-reduced to A. Then the weak hardness notions we will consider here are obtained by different interpretations of the (non)negligible parts of E. For the more recent weak hardness notions of Ambos-Spies and Bakibayev [4] a subclass C is considered to be neglibile if it is contained in some level E_k of the hierarchy E or if it does not contain any sets which are bi-immune to some level E_k (i.e., if for some k no set in C is almost-everywhere 2^{kn}-complex).

Definition 1 (Ambos-Spies and Bakibayev [4]).

(i) A set A is E-nontrivial *if, for any $k \geq 1$, there is a set $B_k \in \mathrm{E} \setminus \mathrm{E}_k$ such that $B_k \leq^p_m A$; and A is* E-trivial *otherwise.*
(ii) A set A is strongly E-nontrivial *if, for any $k \geq 1$, there is an E_k-bi-immune set $C_k \in \mathrm{E}$ such that $C_k \leq^p_m A$; and A is* weakly E-trivial *otherwise.*

Lutz's original weak hardness notion for E in [14] is more technical. It is based on Lutz's resource-bounded measure theory [13] which allows the definition of (pseudo) measures on sufficiently closed complexity classes. This resource bounded measure theory will be used here too in order to define typicalness for complexity classes. More background information on resource-bounded measure theory can be found in the surveys by Lutz [15] and Ambos-Spies and Mayordomo [7] where the latter also explains the relations between resource-bounded measure and randomness to be used here. The following definitions and facts are taken from [7].

Definition 2. *(a) A* martingale *is a real valued function $d : \{0,1\}^* \to [0,\infty)$ such that $d(\lambda) > 0$ and, for every $x \in \{0,1\}^*$, $(d(x0) + d(x1)) \,/\, 2 = d(x)$ holds.*

(b) A martingale d succeeds *on a set A if $\limsup_{n\geq 0} d(A \upharpoonright n) = \infty$ (where $A \upharpoonright n = A(0), \ldots, A(n-1)$ is the initial segment of length n of the characteristic sequence of A). A martingale d* succeeds *on a class* C *if it succeeds on all sets $A \in \mathrm{C}$.*

(c) The (betting) strategy s_d underlying the martingale *d is the function*

$$s_d(x) = \begin{cases} \frac{d(x0)}{2d(x)} & \text{if } d(x) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(d) A $t(n)$-martingale *d is a rational valued martingale $d : \{0,1\}^* \to \mathbb{Q} \cap [0,\infty)$ such that, for the underlying strategy s_d, $s_d \in \mathrm{DTIME}(t(n))$.*

(e) A class C *has* $t(n)$-measure *0 if there is a $t(n)$-martingale which succeeds on* C.

(f) C *has* p-measure *0 (*p_2-measure *0) if C has n^k-measure 0 ($2^{(\log n)^k}$-measure 0) for some $k \geq 1$; and* C *has* computable *or* rec-measure *0 if C has $t(n)$-measure 0 for some computable t. A class* C *has* measure 0 in E *(*EXP, REC*) if* $\mathrm{C} \cap \mathrm{E}$ *(*$\mathrm{C} \cap \mathrm{EXP}$, $\mathrm{C} \cap \mathrm{REC}$*) has p-measure 0 (p_2-measure 0, rec-measure 0). And* C *has* measure 1 in E *(*EXP, REC*) if the complement $\overline{\mathrm{C}}$ of* C *has measure 0 in* E *(*EXP, REC*).*

(g) A set A is $t(n)$-random *if no $t(n)$-martingale succeeds on A (i.e., if the singleton $\{A\}$ does not have $t(n)$-measure 0). A is* p-random *(*p_2-random*) if A is n^k-random ($2^{(\log n)^k}$-random) for all $k \geq 1$ and A is* computably random *or* rec-random *if A is $t(n)$-random for all computable functions $t(n)$.*

As Lutz has shown, the measures on the classes E, EXP and REC are consistent. So a subclass C of E which has measure 0 in E may be considered to be negligible, and if the the class $\mathrm{C}_{\mathcal{P}}$ of sets with a property $\mathcal{P}$ has measure 1 in E then we may say that property $\mathcal{P}$ is typical for the sets in E (and, similarly, for EXP and REC).

Definition 3 (Lutz [14]). *A set A is* measure hard *for* E *if the class of predecessors of A, $\mathrm{P}_{\leq}(A)$, does not have measure 0 in* E *(i.e., if $\mathrm{P}_{\leq}(A) \cap \mathrm{E}$ does not have p-measure 0).*

A final weak hardness notion for E we will consider here is the notion of *category hardness* for E introduced in Ambos-Spies [3] which is based on resource-bounded Baire category (or genericity) in place of resource-bounded measure (or randomness) and which generalizes Lutz's measure hardness. For lack of space we do not introduce this notion here formally but refer the reader to [3] and [7] for details. Here we only note the following implications among the weak hardness notions for E (see Ambos-Spies and Bakibayev [4]):

A measure hard $\Rightarrow A$ category hard $\Rightarrow A$ strongly nontrivial $\Rightarrow A$ nontrivial

In the following we will use the following characterization of resource-bounded measure (and category) in terms of randomness (and genericity) (see [7] and [3]).

Lemma 1. *A class* C *does not have p-measure 0 (p_2-measure 0) if and only if, for any $k \geq 1$, there is an n^k-random ($2^{(\log n)^k}$-random) set in* C. *And* C *does not have rec-measure 0 if and only if, for any computable function $t(n)$, there is a $t(n)$-random set in* C.

By the preceding lemma a set A is measure hard for E if, for any $k \geq 1$, there is an n^k-random set in E which can be reduced to A. The following lemma and variants of this lemma for genericity and bi-immunity in place of randomness yield more simple characterizations of E-measure hardness, E-category hardness and strong E-nontriviality.

Lemma 2 (Ambos-Spies, Terwijn and Zheng [8]). *Let A be an n^2-random set and, for $k \geq 1$, let*

$$A_k = \{x : 0^{k \cdot |x|}x \in A\} \quad \text{and} \quad A'_k = \{x : \, 0^{|x|^{k+1}}x \in A\}. \tag{1}$$

Then $A_k \leq^p_m A$, $A'_k \leq^p_m A$, A_k is n^k-random, and A'_k is $2^{(log\, n)^k}$-random. Moreover, if $A \in \mathrm{E}$ then $A_k \in \mathrm{E}$ too.

Theorem 1 (Characterization Theorems for Weak E-Hardness).

(i) A set A is E*-measure hard if and only if there is an n^2-random set $B \in \mathrm{E}$ such that $B \leq^p_m A$ (Ambos-Spies, Terwijn and Zheng [8]).*
(ii) A set A is E*-category hard if and only if there is an n^2-generic set $B \in \mathrm{E}$ such that $B \leq^p_m A$ (Ambos-Spies [3]).*
(iii) A set A is strongly E*-nontrivial if and only if there is an E_1-bi-immune set $B \in \mathrm{E}$ such that $B \leq^p_m A$ (Ambos-Spies and Bakibayev [4]).*

We conclude our presentation of the basic concepts with some technical result (for a proof see e.g. [7]).

Theorem 2. *Let $t(n)$ be any computable time-bound. Then any $t(n)$-random set is $\mathrm{DTIME}(t(2^n - 1))$-bi-immune. In particular, any n^k-random set is E_k-bi-immune. (Throughout this paper we assume that any computable time-bound $t(n)$ is strictly increasing and time-constructible and satisfies $t(n) \geq n$.)*

3 Noncomputable weakly E-hard sets are untypical

We now turn to the question whether typical sets are weakly hard for E. We first show that, independent of the type of weak hardness we consider, weakly E-hard sets - hence E-hard sets - are rare among all sets. This is a direct consequence of the following more general observation in Ambos-Spies [1].

Theorem 3 (Ambos-Spies [1]). *For any set $A \notin \mathrm{P}$, the upper cone of A under p-m-reducibility, $\mathrm{P}_{\geq}(A) = \{B : A \leq^p_m B\}$, has Lebesgue measure 0.*

So, in particular, by letting A be an E-complete set, the class of E-hard sets has measure 0. In order to get the corresponding result for the weak hardness notions, it suffices to observe that, by countable additivity of Lebesgue measure, Theorem 3 implies that the class of sets which have only the polynomial-time computable sets among their computable predecessors has measure 1.

Corollary 1. *The class $\mathrm{T} = \{A : \mathrm{P}_{\leq}(A) \cap \mathrm{REC} = \mathrm{P}\}$ has measure 1.*

Corollary 2. *The class of the* E*-nontrivial sets has measure 0. Hence, in particular, the classes of the strongly* E*-nontrivial sets, the* E*-category hard sets, the* E*-measure hard sets, and the* E*-hard sets have measure 0.*

Proof. This is immediate by Corollary 1 since any set A in T is E-trivial.

The preceding observations can be effectivized by showing that sufficiently random sets are in the class T hence E-trivial.

Theorem 4. *Let A be computably random and let B be any computable set such that $B \leq^p_m A$. Then $B \in \mathrm{P}$. (In other words, the class of computably random sets is contained in* T*.)*

Proof. For a contradiction assume that $B \notin \mathrm{P}$ and fix f such that $B \leq^p_m A$ via f. Then $f(B)$ is an infinite subset of A. Moreover, since f and B are computable, $f(B)$ is computably enumerable whence $f(B)$ contains an infinite computable set D. So, for a computable function t such that $D \in \mathrm{DTIME}(t(n))$, A is not $\mathrm{DTIME}(t(n))$-immune. It follows by Theorem 2 that A is not $t(n)$-random hence not computably random. But this contradicts the choice of A.

Corollary 3. *Let A be computably random. Then A is* E*-trivial.*

In Section 10 we will show that Corollary 3 (hence Theorem 4) is optimal. Namely, for any computable function $t(n)$ there is an E-nontrivial $t(n)$-random set.

4 E-hard sets are untypical

E-hardness is not only untypical among all sets but, as Mayordomo [16] has shown, also untypical among the sets in E, in EXP, and in REC. This is immediate by the following stronger result.

Theorem 5 (Mayordomo [16]). *The class of* E*-hard sets has p-measure 0.*

Proof (Idea). Berman [9] has shown that no E-hard set is P-(bi-)immune and Mayordomo [16] has shown that the class of P-bi-immune sets has p-measure 1.

Corollary 4 (Mayordomo [16]). *The class of* E*-hard sets has measure 0 in* E*, measure 0 in* EXP*, measure 0 in* REC*, and Lebesgue measure 0.*

Proof. This follows from Theorem 5 since any p-measure-0 class has measure 0 in E, measure 0 in EXP, measure 0 in REC, and classical measure 0.

Note that Mayordomo's theorem has been rephrased in terms of randomness as follows.

Theorem 6 (Ambos-Spies, Terwijn and Zheng [8]). *No n-random set is* E*-hard.*

Proof. By Theorem 2, any n-random set is P-bi-immune, hence (by Berman's observation) not E-hard.

5 Weakly E-complete sets are typical in E

Having seen in the two preceding sections that weakly E-hard sets are untypical among all sets and that E-hard sets are untypical not only among all sets but also among the sets in E, EXP, and REC, we next contrast these observations by a result of Ambos-Spies, Terwijn and Zheng [8] showing that all types of weak hardness for E are typical for sets in E. So a typical set in E is weakly E-complete but not E-complete.

Theorem 7 (Ambos-Spies, Terwijn and Zheng [8]). *The class of* E*-measure hard sets has measure 1 in* E*.*

Proof (Idea). By Theorem 1 (i), any n^2-random set in E is E-measure hard. The claim follows since the class of n^2-random sets has p-measure 1 whence the class of n^2-random sets in E has measure 1 in E.

Corollary 5. *The classes of* E*-nontrivial sets, strongly* E*-nontrivial sets, and* E*-category hard sets have measure 1 in* E*.*

6 E-measure hard and E-category hard sets are untypical in EXP and REC

Having seen in the preceding section that, for any of the weak hardness notions we consider, typical sets in the linear-exponential-time class E are weakly E-hard, we now analyze the question of typicalness of weak E-hardness in the larger polynomial-exponential-time class EXP. Note that, typically, a polynomial-exponential-time set is not a linear-exponential-time set (i.e., more formally, the class

E has measure 0 in the class EXP). So typicalness in E does not imply typicalness in EXP. In fact, Juedes and Lutz [12] have shown that, in contrast to Theorem 7, E-measure hard sets are untypical in EXP, and Ambos-Spies [3] has extended this observation to E-category hardness.

Theorem 8 (Juedes and Lutz [12], Ambos-Spies [3]). *Let A be p-random. Then A is not* E*-category hard, hence not* E*-measure hard.*

Proof (Idea). By Theorem 1 (ii) it suffices to show that there is no n^2-generic set $B \in \mathrm{E}$ such that $B \leq^p_m A$. So, for a contradiction, assume that there is such a set B. Since n^2-generic sets are p-incompressible, it follows from $B \in \mathrm{E}$ and $B \leq^p_m A$ that A is not E-bi-immune. So, by Theorem 2, A is not p-random.

Corollary 6 (Juedes and Lutz [12], Ambos-Spies [3]). *The class of the* E*-category hard sets - hence the class of the* E*-measure hard sets - has measure 0 in* EXP*, measure 0 in* REC*, and classical measure 0.*

Proof (Idea). This follows from Theorem 8 and the fact that the class of p-random sets has p_2-measure 1.

7 Strongly E-nontrivial and E-nontrivial sets are not untypical in EXP

We now contrast the observation that E-category hard sets, hence E-measure hard sets, are untypical among the sets in EXP by showing that strongly E-nontrivial sets - hence E-nontrivial sets - are not untypical in EXP.

Theorem 9. *For any $k \geq 1$ there is a $2^{(\log n)^k}$-random set $A_k \in \mathrm{EXP}$ such that A_k is strongly* E*-nontrivial.*

Corollary 7. *The class of the strongly* E*-nontrivial sets does not have measure* 0 *in* EXP *(hence not p_2-measure 0).*

Proof (of Corollary 7). This follows from Theorem 9 by the characterization of the measure in EXP in terms of randomness.

Proof (of Theorem 9; idea). Fix an n^2-random set $A \in \mathrm{E}$ and, for $k \geq 1$, let $A'_k = \{x : 0^{|x|^{k+1}} x \in A\}$. By Lemma 2, A'_k is $2^{(\log n)^k}$-random and $A'_k \leq^p_m A$. Note that, by the latter, $A'_k \in \mathrm{EXP}$ (since EXP is the downward closure of E under p-m-reducibility). So it suffices to show that A'_k is strongly E-nontrivial. By Theorem 1 (iii), this can be established by showing that there is an E_1-bi-immune set $B \in \mathrm{E}$ such that $B \leq^p_m A'_k$. Such a set B is defined as follows.

For any string z let $l_k(z)$ be the unique number n such that

$$n^{k+1} + n \leq |z| < (n+1)^{k+1} + (n+1)$$

and let $suf_k(z)$ be the last $l_k(|z|)$ bits of z. Then B is defined by

$$B = \{z : suf_k(z) \in A'_k\}.$$

Then, as one can easily check, $B \leq^p_m A'_k$ via suf_k and $B \leq^p_m A$ via $f(z) = 0^{|suf_k(z)|^{k+1}} suf_k(z)$. Since $A \in \mathrm{E}$ and $|f(z)| \leq |z|$ the latter implies that $B \in \mathrm{E}$.

It remains to show that B is E_1-bi-immune. For a contradiction, assume that there is an infinite set $I \in \mathrm{E}_1$ such that, for $z \in I$, $B(z)$ can be computed in $O(2^{|z|})$ steps. By symmetry, w.l.o.g. we may assume that $I \cap B$ is infinite. For f as above let $\hat{I} = \{f(z) : z \in I \cap B\}$. Then, by assumption, $\hat{I}$ is infinite and, by $B \leq^p_m A$ via f, $\hat{I} \subseteq A$. Moreover, it is not hard to shows that $\hat{I} \in \mathrm{E}_4$. So A is not E_4-bi-immune. Since, by Theorem 2, n^4-random sets are E_4-bi-immune this contradicts choice of A.

8 Strongly E-nontrivial and E-nontrivial sets are not typical in EXP

Though, as we have shown in the preceding section, E-nontrivial and strongly E-nontrivial sets are not untypical in EXP, we now show that these sets are not typical in EXP. So these weak notions are neither typical nor untypical among the polynomial-exponential-time sets.

Theorem 10. *For any $k \geq 1$ there is a $2^{(\log n)^k}$-random set $A_k \in \mathrm{EXP}$ such that A_k is* E*-trivial (hence weakly* E*-trivial).*

Corollary 8. *The class of the* E*-trivial sets (hence the class of the weakly* E*-trivial sets) does not have measure* 0 *in* EXP *(hence not p_2-measure 0).*

Theorem 10 is an easy consequence of the following quite deep lemma.

Lemma 3 (Ambos-Spies and Bakibayev [5]). *There is an n^2-random set $A \in \mathrm{EXP}$ which is* E*-trivial.*

Proof (of Theorem 10). Since EXP and the class of the E-trivial sets are closed downwards under $\leq^p_m$ this is immediate by Lemma 3 and by Lemma 2.

9 Strongly E-nontrivial are untypical in REC

Finally, we look at typicalness of the weak hardness notions among the computable sets. By Corollary 6, E-category hard sets hence E-measure hard sets are untypical in REC. So it suffices to consider strong E-nontriviality and E-nontriviality. We first show that strong E-nontriviality is untypical in REC too. Then, in the next section, we will show that E-nontriviality is neither typical nor untypical in REC.

In order to show that the class of strongly E-nontrivial sets has computable measure 0, we first observe that any sufficiently bi-immune set does not have any E_2-bi-immune predecessors in E. Let $\exp_k(n)$ be the k-ply iterated exponential function, i.e., $\exp_0(n) = n$ and $\exp_{k+1}(n) = 2^{\exp_k(n)}$.

Lemma 4. *Let A and B be sets, such that A is* $\mathrm{DTIME}(\exp_5(n))$*-bi-immune,* $B \in \mathrm{E}$ *and* $B \leq_m^p A$*. Then B is not* E_2*-bi-immune.*

Proof (Sketch). Fix $k \geq 1$ such that $B \in \mathrm{E}_k$ and fix a polynomial-time computable function f such that $B \leq_m^p A$ via f. First observe that

$$\forall^\infty x \; (\exp_3(|f(x)|) < |x|). \tag{2}$$

This is shown as follows. For a contradiction assume that (2) fails. Then, for

$$C = \{y : \exists x \; [|x| \leq \exp_3(|y|) \;\&\; y = f(x)]\},$$

C is infinite. Moreover, $C \in \mathrm{DTIME}(\exp_4(n))$ and, for $y \in C$, $A(y)$ can be computed in $\exp_5(|y|)$ steps by computing the least string x of length $< \exp_3(|y|)$ such that $f(x) = y$ and by computing $B(x)$. Note that, by $B \in \mathrm{E}_k$, the latter can be done in $O(2^{k \cdot |x|}) \leq O(2^{k \cdot \exp_3(|y|)}) \leq O(\exp_5(|y|))$ steps. So, contrary to assumption, A is not $\mathrm{DTIME}(\exp_5(n))$-bi-immune.

Now, (2) implies that f compresses B in such a strong way that f is not only not one-to-one but for infinitely many strings x there is a string x' such that $f(x) = f(x')$ and x' is much shorter than x. To be more precise, by (2), the set

$$D = \{x : \exists x' \; (f(x) = f(x') \;\&\; k \cdot |x'| \leq |x|)\}$$

is infinite. Moreover, $D \in \mathrm{E}_2$ and, for $x \in D$, $B(x)$ can be computed in $O(2^{2n})$ steps by computing the least x' such that $f(x) = f(x')$ and by computing $B'(x)$. Note that $B(x') = B(x)$ and, by $B \in \mathrm{E}_k$, $B(x')$ can be computed in $O(2^{k \cdot |x'|}) \leq O(2^{|x|})$ steps. So B is not E_2-bi-immune which completes the proof.

Theorem 11. *Let A be* $\exp_5(n)$*-random. Then A is weakly* E*-trivial.*

Proof. Since, by Theorem 2, any $\exp_5(n)$-random set is $\exp_5(n)$-bi-immune, it follows from Lemma 4 that A does not have any E_2-bi-immune predecessor in E. By (the third part of) Theorem 1 this implies that A is not strongly E-nontrivial.

Corollary 9. *The class of strongly* E*-nontrivial sets has computable measure 0 hence measure 0 in* REC*.*

Proof. This is immediate by Theorem 11 since, for any computable function $t(n)$, the class of $t(n)$-random sets has computable measure 1.

10 E-nontrivial sets are neither typical nor untypical in REC

We conclude our investigation of typicalness of the weak hardness notions for E by showing that (just as in case of EXP) E-nontrivial sets are neither typical nor untypical among the computable sets.

For a proof of the latter we will need the following variant of Lemma 2.

Lemma 5. *Let $t(n)$ be a strictly increasing time constructible function and let A be a computable n-random set. Then, for $A_t = \{z_n : 0^{t(n)} \in A\}$, A_t is computable and $t(n)$-random.*

Theorem 12. *For any strictly increasing, time constructible function $t(n)$ there is a $t(n)$-random set $A_t \in \mathrm{REC}$ such that A_t is* E*-nontrivial.*

Proof. Fix $t(n)$ and let A be any n^2-random set in E. Then, for A_t as in Lemma 5, A_t is computable and $t(n)$-random whence it suffices to show that A_t is E-nontrivial.

By choice of $t(n)$, the tally set $D = \{0^{t(n)} : n \geq 0\}$ is infinite and $D \in \mathrm{P}$. So $A \cap D \leq^p_m A_t$ via f where $f(0^{t(n)}) = z_n$ and, for $x \notin D$, $f(x) = y_0$ for some fixed string $y_0 \notin A_t$. Moreover, since A is n^2-random hence, by Theorem 2, E_1-bi-immune and since $A \in \mathrm{E}$ it follows that $A \cap D \in \mathrm{E} \setminus \mathrm{E}_1$. Since, as shown in Ambos-Spies and Bakibayev [4], any tally set in $\mathrm{E} \setminus \mathrm{E}_1$ is E-nontrivial. It follows that $A \cap D$ is E-nontrivial. Since the class of E-nontrivial sets is closed upwards under $\leq^p_m$, it follows that A_t is E-nontrivial too which completes the proof.

Corollary 10. *The class of computable* E*-nontrivial sets does not have computable measure 0 hence does not have measure 0 in* REC.

It remains to show that the class of the computable E-trivial sets does not have computable measure 0. This is shown by the following minimal pair theorem.

Theorem 13. *Let $B \notin \mathrm{P}$ be computable and let $t : \mathbb{N} \to \mathbb{N}$ be a computable function. There is a computable $t(n)$-random set A such that A and B are a p-m-minimal pair, i.e.,*

$$\forall C \; (C \leq^p_m A, B \Rightarrow C \in \mathrm{P}). \tag{3}$$

The quite involved proof is omitted here. We only note that the proof combines the minimal pair technique of Ambos-Spies [2] with a novel construction of random sets in Ambos-Spies and Kräling [6].

Corollary 11. *For any computable function $t(n)$ there is a computable $t(n)$-random set which is* E*-trivial. Hence the class of computable* E*-trivial sets does not have computable measure 0 and does not have measure 0 in* REC.

Proof. Since, obviously, any set which forms a minimal pair with an E-hard set is E-trivial, it suffices to apply Theorem 13 to an E-complete set B.

11 Summary

The results on typicalness of the weak hardness notions for E can be summarized as follows.

	E	EXP	REC	ALL
E-hard	**untypical**	untypical	untypical	untypical
E-measure hard	**typical**	untypical	untypical	untypical
E-category hard	typical	**untypical**	untypical	untypical
strongly E-nontrivial	typical	neither typical **nor untypical**	**untypical**	untypical
E-nontrivial	typical	**neither typical** nor untypical	**neither typical nor untypical**	**untypical**

Figure 1

Here a property $\mathcal{P}$ is typical (untypical) for the members of a class C if the class of the sets in C with property $\mathcal{P}$ has measure 1 (0) in C.

Since, in Figure 1, the weak hardness notions for E are ordered by decreasing strength, typicalness is preserved downwards and untypicalness is preserved upwards. So the relevant results are the results given in boldface type. Now, for E, the relevant results hold by Corollary 4 due to Mayordomo [16] and Theorem 7 due to Ambos-Spies, Terwijn and Zheng [8]. The relevant results for EXP are Corollary 6 due to Juedes and Lutz [12] and Ambos-Spies [3], Corollary 7, and Corollary 8; and the relevant results for REC are Corollary 9, Corollary 10, and Corollary 11. Finally, the relevant result for ALL is Corollary 2 due to Ambos-Spies [1].

One might expand Figure 1 by considering some classes between EXP and REC like the classes ELEMENTARY and PRIM of the elementary sets and the primitive recursive sets, respectively. For PRIM we obtain the same results as for REC by some straightforward changes of the proofs given there. In case of ELEMENTARY, untypicalness of the strongly E-nontrivial sets and the fact that the E-nontrivial sets are not untypical can be shown as in case of REC. The proof that the E-trivial sets are not untypical among the recursive sets which is based on Theorem 13, however, does not carry over to the class of elementary sets. Namely, as Book has observed (see [2]), no elementary set $A \notin \mathrm{P}$ forms a minimal pair with any E-hard set B. So here the question whether the E-trivial sets are not untypical among the elementary sets remains open.

Finally, note that our investigation in typicalness of the weak hardness notions for E is closely related to the question how much randomness is needed in order to destroy weak hardness. Above we have obtained the following results:

(i) No n-random set is E-hard (Theorem 6).
(ii) No p-random set is E-measure hard or E-category hard (Theorem 8).
(iii) No $\exp_5(n)$-random set is strongly E-nontrivial (Theorem 11).
(iv) No rec-random set is E-trivial (Corollary 3).

Moreover, the results in (ii) and (iv) are optimal since, for any $k \geq 1$, there is an n^k-random set which is E-measure hard hence E-category hard, and, for any computable function $t(n)$ there is a $t(n)$-random set which is E-nontrivial.

The former follows from (the first part of) Theorem 1 and the existence of n^k-random sets in E while the latter holds by Theorem 12. In case of (iii), however, the bound is not optimal. In the proof of Theorem 11 we were very generous when estimating the required upper bounds. So here we leave it as an open problem to find the optimal bound.

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