Weak Completeness Notions for Exponential Time

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Abstract

Lutz [20] proposed the following generalization of hardness: While a problem A is hard for a complexity class C if all problems in C can be reduced to A, Lutz calls a problem weakly hard if a *nonnegligible* part of the problems in C can be reduced to A. For the linear exponential-time class E = DTIME(2^{lin}), Lutz formalized these ideas by introducing a resource-bounded (pseudo) measure on this class and by saying that a subclass of E is negligible if it has measure 0 in E.

Here we introduce and investigate new weak hardness notions for E, called E-*nontriviality* and *strong* E-*nontriviality*, which generalize Lutz's weak hardness notion for E and which are conceptually much simpler than Lutz's concept. Namely, a set A is E-nontrivial if, for any $k \geq 1$, there is a set $B_k \in$ E which can be reduced to A (by a polynomial time many-one reduction) and which cannot be computed in time $O(2^{kn})$, and a set A is strongly E-nontrivial if the sets B_k can be chosen so that they are *almost everywhere* $O(2^{kn})$-complex.

1. Introduction

The standard way for proving a problem to be intractable is to show that the problem is hard or complete for one of the standard complexity classes containing intractable problems. Lutz [20] proposed a generalization of this approach by introducing more general *weak* hardness notions which still imply intractability. While a set A is hard for a class C if *all* problems in C can be reduced to A (by a polynomial-time-bounded many-one reduction) and complete if it is hard and a member of C, Lutz proposed to call a set A weakly hard if a *nonnegligible* part of C can be reduced to A and to call A weakly complete if in addition $A \in$ C. For the exponential-time classes E = DTIME(2^{lin}) and EXP = DTIME(2^{poly}), Lutz formalized these ideas by introducing resource-bounded (Lebesgue) measures on these classes and by saying that a subclass of E is negligible if it has measure 0 in E (and similarly for EXP). A variant of these concepts, based on resource-bounded Baire category in place of measure, was introduced by Ambos-Spies [4] where now a class is declared to be negligible if it is meager in the corresponding resource-bounded sense.

A certain drawback of these weak hardness notions in the literature, called measure-hardness and category-hardness in the following, is that they are based on the somewhat technical concepts of resource-bounded measure and resource-bounded category, respectively. So here we introduce some alternative weak hardness notions which are conceptually much simpler and are solely based on the basic concepts of computational complexity theory.

While the weak hardness notions in the literature implicitly used the fact that, by the time-hierarchy theorem, the linear-exponential time class E is a proper hierarchy of time complexity classes, where the individual levels are the deterministic-time classes $\mathrm{E}_k = \mathrm{DTIME}(2^{kn})$ $(k \geq 1)$, our primary new weak hardness notion for E, called E-*nontriviality*, is based on this observation explicitly. We consider a subclass of E to be negligible, if it is contained in a fixed level E_k of the linear-exponential-time hierarchy. In other words, a set A is E-nontrivial if it has predecessors (under p-m-reducibility) from arbitrarily high levels of this hierarchy.

Since any level E_k of E has measure 0 in E and is meager in E, E-nontriviality generalizes the previously introduced weak hardness notions for E. On the other hand, since $\mathrm{P} \subseteq \mathrm{E}_1$, E-nontriviality still guarantees intractability. In fact, we may argue that E-nontriviality is the most general weak hardness notion for E if for such a notion we do not only require that it generalizes E-hardness and implies intractability but that it also reflects the internal, hierarchical structure of E.

The second new weak hardness notions we will consider, called *strong* E-*nontriviality*, may be viewed as a link between the weak notion of E-nontriviality and the stronger notions of measure- and category-hardness. A strongly E-nontrivial set does not only have predecessors from arbitrarily high levels $\mathrm{E} \setminus \mathrm{E}_k$ of the hierarchy E but, for any given $k \geq 1$, it has a predecessor in E which is *almost-everywhere* complex for the kth level of the linear exponential-time hierarchy, i.e., which is E_k-bi-immune.

The outline of the paper is as follows.

After formally introducing our new weak hardness notions for E - E-nontriviality and strong E-nontriviality - in Section 2, in Section 3 we give some examples of intractable but still E-trivial sets in E thereby showing that E-nontriviality is a strict refinement of intractability. First we observe that sets of sufficiently low hyper-polynomial complexity are E-trivial. Though this observation is not surprising, it gives us some first nontrivial facts on the distribution of the E-trivial and the E-nontrivial sets in E. For instance it implies that the only sets which code all E-trivial sets in E are the E-hard sets. We then show (what might be more surprising) that there are E-trivial sets in E of arbitrarily high complexity, i.e., that, for any $k \geq 1$, there is an E-trivial set in $\mathrm{E} \setminus \mathrm{E}_k$. In fact, by generalizing a result of Buhrman and Mayordomo [13] for measure hardness, we obtain some natural examples of such sets by showing that the sets of the Kolmogorov-random strings w.r.t. exponential-time-bounded computations are E-trivial.

In Section 4 we give some examples of (strongly) E-nontrivial sets and give a separation theorem for the weak E-completeness notions mentioned before. In order to show that there are strongly E-nontrivial sets in E which are not category and measure complete for E and that there are E-nontrivial sets which are not strongly E-nontrivial we analyse the minimum density of the the complete sets under the various weak hardness notions considered here. In particular, we show that there are tally strongly E-nontrivial sets in E whereas no category complete (hence no measure complete) set for E has this property, and that there are exptally E-nontrivial sets in E whereas no strongly E-nontrivial set in E has this property.

In Section 5 we analyse the information content of weakly complete sets thereby giving some more structural differences among the complete sets under the various weak hardness notions. For instance we show, that the effective disjoint union of two E-trivial sets is E-trivial again, and that E-trivial sets *don't help.* The latter means that if an E-hard set H can be reduced to the effective disjoint union of sets A and B where A is E-trivial then H can be reduced to B already, i.e., B is E-hard. In other words, if we decompose an E-complete set into two incomplete parts then both parts are E-nontrivial. For the other weak hardness notions the corresponding facts fail. In fact, any set $A \in \mathrm{E}$ can be split into two sets which are not strongly E-trivial.

Finally, in Section 6 we give a short summary of results on some other aspects of our new weak hardness notions which will be presented in more detail elsewhere.

We conclude this section by introducing some notation and by summarizing some basic definitions and facts related to the exponetial time classes to be used in the following.

Our notation is standard. Unexplained notation can be found in the monographs of Balcázar et al. [11] and [12]. We let $\Sigma^* = \{0, 1\}^*$ be the set of (finite binary) strings. A *set* or *problem* or *language* is a subset of Σ^*, a *class* is a set of sets, i.e., a subset of the power set of Σ^*. Strings will be denoted by lower case letters from the end of the alphabet (u, $\ldots$, z, u_n, $\ldots$), sets by italic capital letters (A, B, C, $\ldots$, A_n, $\ldots$), and classes by straight capital letters (A, B, C, $\ldots$, A_n, $\ldots$). The $(n+1)$th string with respect to the length-lexicographical ordering $\leq$ is denoted by z_n, i.e., $z_0 = \lambda, z_1 = 0, z_2 = 1, z_2 = 00, \ldots$. Note that $|z_n| \approx \log(n)$ where $|x|$ denotes the length of x. For a set A and string x we write $A(x) = 1$ if $x \in A$ and $A(x) = 0$ if $x \notin A$.

The exponential time classes we will deal with are the classes

$$\mathrm{E} = \bigcup_{k \geq 1} \mathrm{DTIME}(2^{kn}) \qquad (\textit{Linear Exponential Time}) \tag{1}$$

$$\mathrm{EXP} = \bigcup_{k \geq 1} \mathrm{DTIME}(2^{n^k}) \qquad (\textit{Polynomial Exponential Time}) \tag{2}$$

where we will use the following abbreviations for the individual levels of these classes:

$$\mathrm{E}_k = \mathrm{DTIME}(2^{kn}) \text{ and } \mathrm{EXP}_k = \mathrm{DTIME}(2^{n^k}).$$

Note that, by the time-hierarchy theorem, the hierarchies of the linear-exponential-time classes and of the polynomial-exponential-time classes are proper, and that

$$\mathrm{E}_1 = \mathrm{EXP}_1 \subset \mathrm{E} \subset \mathrm{EXP}_2.$$

For comparing the complexity of problems we use the polynomial-time-bounded version of many-one reducibility (*p*-*m*-reducibility for short) where a set B is *p*-*m*-reducible to a set A ($B \leq^p_m A$) via f if f is polynomial-time computable function $f : \Sigma^* \to \Sigma^*$ and $B(x) = A(f(x))$ holds for all strings x. Note that $\leq^p_m$ is a preordering, i.e., reflexive and transitive. So *p-m-equivalence* is an equivalence relation where sets A and B are p-m-equivalent ($A =^p_m B$) if $A \leq^p_m B$ and $B \leq^p_m A$ hold.

We call B a *predecessor* of A (and A a *successor* of B) if B is p-m-reducible to A, and we let $\mathrm{P}_m(A) = \{B : B \leq^p_m A\}$ denote the class of predecessors of A under p-m-reducibility. A set A is *hard* for a class C (or C-*hard* for short) if $\mathrm{C} \subseteq \mathrm{P}_m(A)$, i.e., if any set in C is p-m-reducible to A, and A is *complete* for a class C (or C-*complete* for short) if A is a member of C and C-hard.

The (downward) closure of a class C under p-m-reducibility is the class of the predecessors of the members of C,

$$Cl^p_m(\mathrm{C}) = \bigcup_{A \in \mathrm{C}} \mathrm{P}_m(A),$$

and a class $\mathcal{C}$ is *closed* under p-m-reducibility if $Cl^p_m(\mathrm{C}) = \mathrm{C}$ holds.

Note that EXP is closed under p-m-reducibility, i.e., $Cl^p_m(\mathrm{EXP}) = \mathrm{EXP}$. The other exponential classes $\mathrm{E}, \mathrm{E}_k, \mathrm{EXP}_k$, however, are not closed under p-m-reducibility. This is a consequence of the well known *Padding Lemma.*

Lemma 1.1 (Padding Lemma). *For any set $A \in \mathrm{E}_k$, the set $A_k = \{0^{k\cdot|x|}1x : x \in A\}$ is in E_1 and p-m-equivalent to A, and, for any set $A \in \mathrm{EXP}_k$, the set $\hat{A}_k = \{0^{|x|^k}1x : x \in A\}$ is in E_1 and p-m-equivalent to A.*

So

$$Cl^p_m(\mathrm{E}_k) = Cl^p_m(\mathrm{E}) = Cl^p_m(\mathrm{EXP}_k) = Cl^p_m(\mathrm{EXP}) = \mathrm{EXP}$$

(for all $k \geq 1$). Though E is not closed under p-m-reducibility, we will use the observation that E is closed under p-m-reductions of linearly bounded size.

Lemma 1.2. *Assume that $A \in \mathrm{E}_k$ and $B \leq^p_m A$ via f where $|f(x)| \leq k' \cdot |x| + k''$ for all x ($k, k' \geq 1, k'' \geq 0$). Then $B \in \mathrm{E}_{k\cdot k'}$.*

Proof. Given a string x of length n, $B(x)$ can be computed using the identity $B(x) = A(f(x))$ where, by $|f(x)| \leq k' \cdot n + k''$ and by $A \in \mathrm{DTIME}(2^{kn})$, $A(f(x))$ can be computed in $O(2^{k\cdot(k'\cdot n + k'')}) \leq O(2^{k\cdot k'})$ steps. □

2. E-Nontriviality and Strong E-Nontriviality

We now introduce the two new central notions which we will study in this paper. As pointed out in the introduction these notions have been inspired by Lutz's idea of generalizing hardness and completeness for a complexity class C $\supset$ P in such a way that hardness still guarantees intractability. Following Lutz [20], we call a problem A weakly hard for a complexity class C if a *nonnegligible* part of the problems in C can be p-m-reduced to A, and we call A *weakly complete* if $A \in$ C and A is weakly hard for C. In order that this notion meets its goals, the family of the nonnegligible subclasses of C must have the following properties:

(i) The class C itself has to be nonnegligible thereby guaranteeing that any hard (complete) set for C (under p-m-reducibility) is weakly hard (weakly complete).

(ii) The class P of the polynomial time computable sets has to be negligible thereby guaranteeing that weakly hard problems for C are intractable.

Moreover, it is natural to ask that subclasses of negligible classes are negligible again and that finite unions of negligible classes remain negligible:

(iii) For any subclasses $\mathrm{C}', \mathrm{C}''$ of C such that $\mathrm{C}' \subseteq \mathrm{C}''$ and C'' is negligible, C' is negligible too.

(iv) For negligible subclasses $\mathrm{C}', \mathrm{C}''$ of C, $\mathrm{C}' \cup \mathrm{C}''$ is negligible too.

Finally, the definition of negligibility should reflect the structure of C.

For the linear exponential-time class E = DTIME(2^{lin}) (and for other sufficiently closed complexity classes like the polynomial exponential-time class EXP), Lutz formalized these ideas by introducing a resource-bounded (pseudo) measure on this class and by saying that a subclass of E is negligible if it has measure 0 in E. He then showed that the above conditions (i) - (iv) are satisfied whence the corresponding weak hardness and completeness notions are meaningful. He also showed that his negligibility notion reflects the hierarchical structure of E as follows.

(v) For any $k \geq 1$, E_k is negligible.

I.e. a class which intersects only finitely many of the infinitely many levels of $\mathrm{E} = \bigcup_{k\geq 1} \mathrm{E}_k$ is negligible.

In the following we refer to the weak hardness (completeness) notion of Lutz [20] which is based on the measure on E as *measure hardness (measure completeness) for* E or E-*measure hardness* (E-*measure completeness*) for short.

Following Lutz [20], Ambos-Spies [4] introduced alternative weak hardness notions for E which are based on Baire category in place of measure. Correspondingly, the negligible subclasses of E are now those which are meager in E. In fact, there are several Baire category concepts for E discussed in [4], but one of the concepts - called AFH-category there - proved to be of particular interest since it is compatible with measure, namely AFH-meager subclasses of E have measure 0 in E. So the corresponding weak hardness concept is more general than Lutz's measure hardness.

In the following we refer to the weak hardness (completeness) notion of Ambos-Spies [4] based on AFH-category on E as *category hardness (category completeness) for* E or E-*category hardness* (E-*category completeness*) for short.

Though the weak hardness and completeness notions of Lutz [20] and Ambos-Spies [4] are very intuitive in being based on two of the classical systems for measuring the size of a class, namely Lebesgue measure and Baire category, a certain drawback of these notions is that the resource-bounded measure and category notions they are based on are somewhat technical. (For this reason we also do not introduce these concepts here formally.) So here we propose alternative weak hardness notions for E which are solely based on standard concepts of complexity theory.

By our above analysis, the most general negligibility notion for E which reflects the structure of E (i.e., satisfies (v) above) is obtained by calling a subclass of E negligible if it is contained in one of the levels E_k of E. In other word, a subclass C of E is nonnegligible if it has members from arbitrary high levels of the linear exponential hierarchy E. We call the corresponding weak hardness notion E-*nontriviality.*

Definition 2.1. *A set A is* trivial for E *(or* E-trivial *for short) if*

$$\exists\, k \geq 1\ (\mathrm{P}_m(A) \cap \mathrm{E} \subseteq \mathrm{E}_k) \tag{3}$$

holds, and A is nontrivial for E *(or* E-nontrivial *for short) otherwise, i.e., if*

$$\forall\, k \geq 1\ \exists\, B \in \mathrm{E} \setminus \mathrm{E}_k\ (B \leq^p_m A) \tag{4}$$

holds.

The second concept we will consider here is a strengthening of E-nontriviality and is called *strong* E-*nontriviality.* While, for an E-nontrivial set A, we require that, for any $k \geq 1$, there is a set B in E which can be p-m-reduced to A and which is *infinitely often* 2^{kn}*-complex* (i.e., any algorithm computing B requires more than $2^{k|x|}$ steps for infinitely many inputs x), for a strongly E-nontrivial set A we require that there is such a set B which is *almost-everywhere* $2^{k|x|}$*-complex* (i.e., any algorithm computing B requires more than $2^{k|x|}$ steps for all but finitely many inputs x). Since almost everywhere complexity coincides with bi-immunity, i.e., since a set A is a.e. $t(n)$-complex if and only if A is DTIME($t(n)$)-bi-immune (see Balcázar et al. [12]), we formally define strong E-nontriviality in terms of bi-immunity. Recall that a set A is C-*bi-immune* for a class C if there is no infinite set $B \in \mathrm{C}$ such that $B \subseteq A$ or $B \cap A = \emptyset$.

Definition 2.2. *A set A is* strongly nontrivial for E *(or* strongly E-nontrivial *for short) if*

$$\forall\, k \geq 1\ \exists\, B \in \mathrm{P}_m(A) \cap \mathrm{E}\ (B \textit{ is } \mathrm{E}_k\textit{-bi-immune}) \tag{5}$$

holds; and A is weakly trivial for E *(or* weakly E-trivial *for short) otherwise.*

Note that, for any E_k-bi-immune set B, $B \notin \mathrm{E}_k$. So, any strongly E-nontrivial set A is E-nontrivial. Also note that strong E-nontriviality may be viewed as weak hardness for E if we call a subclass C of E nonnegligible if it contains E_k-bi-immune sets for all $k \geq 1$. That this negligibility notion satisfies (i) above follows for instance from the time-hierarchy theorem for a.e. complexity by Geske et al. [14] which implies that there are E_k-bi-immune sets $A \in E_{k+1}$ (for any $k \geq 1$). In fact, as shown by Ambos-Spies [4], any E-category hard set has predecessor in E which are E_k-bi-immune (for all $k \geq 1$). So E-category hard sets hence E-measure hard sets are strongly E-nontrivial. So we obtain the following relations among the weak hardness notions for E.

Lemma 2.3. *For any set A the following hold.*

$$\begin{array}{c} A \text{ E-}\textit{hard} \\ \Downarrow \\ A \text{ E-}\textit{measure hard} \\ \Downarrow \\ A \text{ E-}\textit{category hard} \\ \Downarrow \\ A \textit{ strongly } \text{E-}\textit{nontrivial} \\ \Downarrow \\ A \text{ E-}\textit{nontrivial} \\ \Downarrow \\ A \textit{ intractable} \end{array} \tag{6}$$

PROOF. The first implication has been shown in Lutz [20], while, as pointed out before, the second and third implications are shown in Ambos-Spies [4]. The fourth implication follows from the fact that no set in E_k is E_k-bi-immune. Finally, the last implication follows from the fact that $\mathrm{P} \subset E_1$. □

Lutz [20] and Ambos-Spies [4] have shown that the first three inplications in Lemma 2.3 are strict, even if we consider only sets in E (i.e., the corresponding weak completeness notions). In the following sections we will show that the other implications are strict too.

We conclude this section with some simple observations on the E-nontrivial and strongly E-nontrivial sets.

Note that the class of the (strongly) E-nontrivial sets is closed upwards under $\leq^p_m$ hence, in particular, closed under p-m-equivalence. By definition, an E-nontrivial set A has predecessors in infinitely many levels $\mathrm{E}_{k+1} \setminus \mathrm{E}_k$ of the linear exponential hierarchy E. In fact, an E-nontrivial set has predecessors in all of these levels.

Lemma 2.4. *Let A be* E-*nontrivial. Then*

$$\forall\, k \geq 1\ \exists\, B \in \mathrm{E}_{k+1} \setminus \mathrm{E}_k\ (B \leq^p_m A) \tag{7}$$

holds.

Lemma 2.4 directly follows from the following variation of the Padding Lemma which is obtained by calibrating the amount of padding.

Lemma 2.5 (Second Padding Lemma). *Let A and $k \geq 1$ be given such that $A \in \mathrm{E}_{k+1} \setminus \mathrm{E}_k$. Then, for any $k' \leq k$ (with $k' \geq 1$), there is a set $A' \in \mathrm{E}_{k'+1} \setminus \mathrm{E}_{k'}$ such that $A' =^p_m A$.*

Proof (Sketch). Given $k \geq 2$ and $A \in \mathrm{E}_{k+1} \setminus \mathrm{E}_k$, let $A' = \{0^{f(|x|)}1x : x \in A\}$ for $f(n) = \lfloor \frac{n}{k} \rfloor$. Then, as one can easily show, $A' =^p_m A$ and $A' \in \mathrm{E}_k \setminus \mathrm{E}_{k-1}$. The claim follows by induction. □

For strongly E-nontrivial sets we can make the correspondig observation. By definition, for a strongly E-nontrivial set A there are infinitely many k such that A has a predecessor $B \in \mathrm{E}$ which is E_k-bi-immune but not E_{k+1}-bi-immune. By applying the Second Padding Lemma above one may argue that this is in fact true for all k:

$$\forall\, k \geq 1\ \exists B \in \mathrm{E} \cap \mathrm{P}_m(A)\ (B \text{ is } \mathrm{E}_k\text{-bi-immune \& } B \text{ is not } \mathrm{E}_{k+1}\text{-bi-immune}) \tag{8}$$

We omit the proof and rather give another alternative characterization of strong E-nontriviality which is of greater technical interest.

Theorem 2.6 (Characterization Theorem for Strong Nontriviality). *A set A is strongly* E*-nontrivial if and only if there is an* E_1*-bi-immune set* $B \in \mathrm{E}$ *such that* $B \leq^p_m A$.

The nontrivial direction of Theorem 2.6 follows from the following lemma by considering the sets B_k there for a given E_1-bi-immune predecessor B of A in E.

Lemma 2.7. *Let B be* E_1*-bi-immune. Then, for any $k \geq 1$, there is an* E_k*-bi-immune set B_k and an* EXP_k*-bi-immune set B'_k such that $B_k, B'_k \in \mathrm{P}_m(B)$. If moreover $B \in \mathrm{E}$ then the set B_k can be chosen such that $B_k \in \mathrm{P}_m(B) \cap \mathrm{E}$.*

Proof. The idea is taken from Ambos-Spies et al. [10] where a similar lemma for randomness in place of bi-immunity is proven. So we only sketch the proof.

Let $B_k = \{x : 0^{k|x|}1x \in B\}$ and $B'_k = \{x : 0^{|x|^k}1x \in B\}$. Then $B_k \leq^p_m B$ via $f(x) = 0^{k|x|}1x$ and $B'_k \leq^p_m B$ via $g(x) = 0^{|x|^k}1x$. Moreover, if $B \in \mathrm{E}$, say $B \in \mathrm{E}_m$, then $B_k \in \mathrm{E}_{(k+1)m}$, hence $B_k \in \mathrm{E}$ too.

It remains to show that B_k is E_k-bi-immune and B'_k is EXP_k-bi-immune. We will prove the former, the proof of the latter is similar.

For a contradiction assume that B_k is not E_k-bi-immune. By symmetry, we may assume that there is an infinite set $C' \subseteq B_k$ such that $C' \in \mathrm{E}_k$.

Now let $C = \{0^{k|x|}1x : x \in C'\}$. Then, by infinity of C', C is infinite, and, by $C' \subseteq B_k$ and by definition of B_k, $C \subseteq B$. Moreover $C \in \mathrm{E}_1$, since, for a string y we can decide whether $y \in C$ by first checking (in polynomial time) whether there is a string x such that $y = 0^{k|x|}1x$ and, if so, by checking in $O(2^{k|x|}) \leq O(2^{|y|})$ steps whether $x \in C'$. So B is not E_1-bi-immune contrary to assumption. □

3. Some Examples of E-Trivial Sets in E

In order to show that E-nontriviality does not coincide with intractability, here we give some examples of intractable but E-trivial sets. As one might expect, sets of sufficiently low time complexity are E-trivial. As we will also show, however, E-trivial sets can be found at all levels of the linear-exponential hierarchy.

Lemma 3.1. *Let t be a nondecreasing, time constructible function such that, for some number $k \geq 1$,*

$$t(p(n)) \leq_{a.e.} 2^{kn} \tag{9}$$

for all polynomials p. Then any set $A \in \mathrm{DTIME}(t(n))$ is E*-trivial.*

Proof. Given $A \in \mathrm{DTIME}(t(n))$ it suffices to show that $\mathrm{P}_m(A) \subseteq \mathrm{E}_k$. So fix $B \in \mathrm{P}_m(A)$, let f be a polynomial time computable function f such that $B \leq_m^p A$ via f, and let p be a polynomial time bound for f. Then $B(x)$ can be computed by using the identity $B(x) = A(f(x))$ in $O(p(|x|) + t(p(|x|))$ steps since it takes at most $p(|x|)$ steps to compute $f(x)$ and, by $A \in \mathrm{DTIME}(t(n))$ and by $|f(x)| \leq p(|x|)$, it takes at most $t(p(|x|))$ steps to compute $A(f(x))$ for the given string $f(x)$. So, by (9), $B(x)$ can be computed in $O(2^{kn})$ steps, i.e., $B \in \mathrm{E}_k$. □

Theorem 3.2. *There is an* E*-trivial set $A \in \mathrm{E} \setminus \mathrm{P}$.*

Proof. By Lemma 3.1 it suffices to show that there is a nondecreasing time constructible function t such that $\mathrm{P} \subset \mathrm{DTIME}(t(n))$ and such that $t(p(n)) \leq_{a.e.} 2^n$ for all polynomials p.

Note that, for any polynomial p,

$$p(n) \leq_{a.e.} 2^{(\log n)^2} \leq_{a.e.} 2^{(\log n)^4} \text{ and } 2^{(\log n)^4} \notin O(2^{(\log n)^2} \cdot \log(2^{(\log n)^2})).$$

So $\mathrm{P} \subseteq \mathrm{DTIME}(2^{(\log n)^2}) \subset \mathrm{DTIME}(2^{(\log n)^4})$ where strictness of the latter inclusion holds by the time-hierarchy theorem. Moreover, $2^{(\log\ p(n))^4} \leq_{a.e.} 2^n$ for all polynomials p. So the nondecreasing and time constructible function $t(n) = 2^{(\log n)^4}$ has the required properties. □

Theorem 3.2 can be strengthened by using some results on the p-m-degrees of hyperpolynomial shifts proven in Ambos-Spies [1]. Here a set $A_h = \{1^{h(|x|)}0x : x \in A\}$ is called a *hyperpolynomial shift* of A if h is a time constructible, nondecreasing function $h : \mathbb{N} \to \mathbb{N}$ such that h dominates all polynomials. Note that, for any hyperpolynomial shift A_h of a set $A \in \mathrm{EXP}$, $A_h \in \mathrm{DTIME}(t(n))$ for a function $t(n)$ as in Lemma 3.1 whence A_h is E-trivial. Now, in [1] it has been shown that,

(i) for any set $A \notin P$ and for any hyperpolynomial shift A_h of A, A_h is a strict predecessor of A, i.e., $A_h <_m^p A$ and

(ii) for any computable sets A and B such that $A \not\leq_m^p B$ there is a hyperpolynomial shift A_h of A such that $A_h \not\leq_m^p B$.

These results imply the following facts on the distribution of the E-trivial sets in E w.r.t. p-m-reducibility. By letting A be any E-complete set, the above implies:

Theorem 3.3. *(a) For any set $A \in \mathrm{E} \setminus \mathrm{P}$ there is an* E*-trivial set $T \in \mathrm{E} \setminus \mathrm{P}$ such that $T <^p_m A$.*

(b) For any computable set B which is not E*-hard there is an* E*-trivial set T such that $T \not\leq^p_m B$.*

Proof. As observed above, for any set $A \in \mathrm{E}$ and any hyperpolynomial shift A_h of A, $A_h \in \mathrm{E}$, A_h is E-trivial, and (by (i)) $A_h <^p_m A$. So for a proof of (a) it suffices to show that for given $A \in \mathrm{E} \setminus \mathrm{P}$ there is a hyperpolynomial shift A_h of A such that $A_h \notin \mathrm{P}$. But this follows from (ii) above by letting B be any polynomial time computable set. Similarly, for a proof of (b) it suffices to show that there is a set $A \in \mathrm{E}$ which possesses a hyperpolynomial shift A_h such that $A_h \not\leq^p_m B$. But this follows from (ii) too by letting A be any E-complete set. □

Note that, by part (a) of Theorem 3.3 any intractable set in E bounds an intractable E-trivial set in E while, by part (b), the only sets in E which bound all the E-trivial sets in E are the E-complete sets.

Having given examples of intractable E-trivial sets of low hyperpolynomial complexity, we now show that there are E-trivial sets at arbitrarily high levels $\mathrm{E} \setminus \mathrm{E}_k$ of the E-hierarchy. (So, by Lemma 2.5, there are E-trivial sets at all levels $\mathrm{E}_{k+1} \setminus \mathrm{E}_k$ of the E-hierarchy.)

Theorem 3.4. *For any $k \geq 1$ there is an* E*-trivial set A in $\mathrm{E} \setminus \mathrm{E}_k$.*

The idea of the proof is as follows. Given $k \geq 1$, by a diagonalization argument we construct a set $A \in \mathrm{E}_{k+2} \setminus \mathrm{E}_k$ such that any set B which is p-m-reducible to A will be p-m-reducible to A via a polynomial-time computable function f such that $|f(x)| \leq 2|x|$. Then, by Lemma 1.2, $\mathrm{P}_m \subseteq \mathrm{E}_{2(k+2)}$. So A is E-trivial.

To be more precise, we will apply the following lemma generalizing Lemma 1.2.

Lemma 3.5 (Boundedness Lemma). *Let A and B be sets and let f be a polynomial time computable function such that $A \in \mathrm{E}_k$, $B \leq^p_m A$ via f, and*

$$\forall\, x\ (|f(x)| \leq k' \cdot |x| + k'' \quad or \quad f(x) \notin A) \tag{10}$$

(for some $k, k' \geq 1, k'' \geq 0$). Then $B \in \mathrm{E}_{k' \cdot k}$.

Proof. Using the identity $B(x) = A(f(x))$ we can compute $B(x)$ for a given string x of length n in $O(2^{(k' \cdot k)n})$ steps as follows. First, in $O(poly(n))$ steps compute $f(x)$. Then, in $O(poly(n))$ steps check whether $|f(x)| \leq k' \cdot n + k''$ or not. If not, then $B(x) = A(f(x)) = 0$. If so, by $A \in \mathrm{E}_k$, compute $B(x) = A(f(x))$ in $O(2^{k \cdot |f(x)|}) \leq O(2^{k \cdot (k' \cdot n + k'')}) = O(2^{(k' \cdot k)n})$ steps. □

Proof of Theorem 3.4. Fix $k \geq 1$ and let $\{E^k_e : e \geq 0\}$ and $\{f_e : e \geq 0\}$ be enumerations of E_k and of the class of the polynomial time computable functions, respectively, such that $E^k_e(x)$ can be computed in time $O(2^{(k+1) max(e,|x|)})$ and $f_e(x)$ can be computed in time $O(2^{max(e,|x|)})$ (uniformly in e and x).

By a diagonal argument we define a set $A \in \mathrm{E}_{k+2}$ which meets the requirements

$$\Re_{2e} : A \neq E^k_e$$

and

$$\Re_{2e+1} : \forall x \in \Sigma^* \ (|f_e(x)| > |x| + e + 1 \Rightarrow f_e(x) \notin A)$$

for $e \geq 0$.

Obviously, the requirements with even indices ensure that $A \notin \mathrm{E}_k$. Similarly, by $A \in \mathrm{E}_{k+2}$, the requirements with odd indices ensure that A is E-trivial since, by Lemma 3.5, $\mathrm{P}_m(A) \subseteq \mathrm{E}_{k+2}$.

For the definition of A, call a string y *forbidden* if $y = f_e(x)$ for some number e and some string x such that $|x| + e + 1 < |y|$, and let F be the set of forbidden strings. Then, in order to meet the requirements $\Re_{2e+1}$ $(e \geq 0)$ it suffices to ensure that no forbidden string is put into A, i.e., to enusre that $A \cap F = \emptyset$.

Note that the number $f(n)$ of pairs (x, e) such that $|x| + e + 1 < n$ is less than 2^n. Namely, for $n \leq 1$, $f(n) = 0$ and, for $n \geq 2$,

$$\begin{array}{rcl}
f(n) & = & |\{(x,e) : |x| + e + 1 < n\}| \\
 & = & \sum_{e=0}^{n-2} |\{0,1\}^{<n-e-1}| \\
 & = & \sum_{e=0}^{n-2} (2^{n-e-1} - 1) \\
 & = & \sum_{e=1}^{n-1} (2^e - 1) \\
 & < & \sum_{e=0}^{n-1} (2^e) \\
 & = & 2^n - 1
\end{array}$$

So the question of whether a string y of lenght n is forbidden can be decided in $O(2^{2n})$ steps since it suffices to compute for each of the $f(n) < 2^n$ pairs (x, e) statisfying $|x|+e+1 < n$ the value of $f_e(x)$ which, by choice of the functions f_e can be done in $O(2^n)$ steps. So $F \in \mathrm{E}_2$. Moreover, since $f(n)$ is a bound on the number of forbidden strings of length n, at least one of the 2^n strings of length n is not forbidden, i.e, $\overline{F} \cap \{0,1\}^n \neq \emptyset$ for all $n \geq 0$.

So if we let $A = \{y_e : y_e \notin E_e^k\}$ where y_e is the least string of length e which is not forbidden then all requirements are met and, as one can easily check, $A \in \mathrm{E}_{k+2}$. □

Alternatively we obtain E-trivial sets of high complexity in E by refining a result of Buhrman and Mayordomo [13] on the random strings in the setting of time-bounded Kolmogorov complexity. Call a string x $t(n)$-K-random if there is no $t(n)$-time-bounded Turing machine compressing x (for the formal definition, see e.g. Li and Vitanyi [18]), and let R^t be the set of $t(n)$-K-random strings. Buhrman and Mayordomo have shown that, for an exponential time bound $t(n) = 2^{kn}$ $(k \geq 2)$, the set R^t of $t(n)$-K-random strings is not weakly E-complete in the sense of Lutz. This can be strengthened as follows.

Theorem 3.6. *For $t(n) = 2^{kn}$ $(k \geq 2)$, the set R^t of the $t(n)$-K-random strings is* E-*trivial.*

Proof (Idea). As one can easily check, $R^{2^{kn}} \in$ E. So, given a set A and a p-m-reduction function f such that $A \leq^p_m R^{2^{kn}}$ via f, by Lemma 3.5 it suffices to show that for almost all x such that $|f(x)| > 2|x|$, $f(x) \notin R^{2^{kn}}$. But this is straightforward, since for a string x with $|f(x)| > 2|x|$, $f(x)$ can be computed from the shorter string x in polynomial time hence, for sufficiently large x, in time $2^{k|x|}$. □

Remark. In this paper we only look at E-(non)trivial sets in E, i.e., investigate (strong) E-nontriviality as a weak completeness notion, and do not consider the corresponding, more general weak hardness notion. So we only remark here that outside of E we can find computable E-trivial sets of arbitrarily high time complexity. Moreover, there are numerous noncomputable E-trivial sets. In fact, the class of E-trivial sets has (classical) Lebesgue measure 1. These observations immediately follow from results in the literature about p-m-minimal pairs. (Sets A and B form a p-m-minimal pair if, for any set C such that $C \leq^p_m A$ and $C \leq^p_m B$, $C \in$ P.) It suffices to observe that, for sets A and B such that B is E-hard and A and B form a minimal pair, A is E-trivial.

xxxxxx eventuell erst im abschnitt conclusion xxxxxxxxxxx

4. On the Density of E-Nontrivial and Strongly E-Nontrivial Sets

Lutz [20] has shown that any E-measure complete set is exponentially dense whereas for E-category completeness introduced by Ambos-Spies [4] there are sparse E-category complete sets. (In fact, in [4] weak completeness notions were introduced for various time-bounded category concepts. Here we refer to the category concept called AFH-category there which proved to be useful for analysing time-bounded measure (see [7] and [10] for more details) since - in contrast to the classical Baire category concept - this concept is compatible with measure.) By results in [4] and [10], however, E-category complete sets cannot be tally.

So, in order to distinguish strong E-nontriviality from E-category completeness (and E-measure completeness), it suffices to show that there are tally strongly E-nontrivial sets in E. To do so we need the following observation on E_{k+1}-bi-immune sets.

Lemma 4.1. *Let $A \in$ E be E_{k+1}-bi-immune ($k \geq 1$) and let $\hat{A}$ be the length language $\hat{A} = \{x : 0^{|x|} \in A\}$. Then $\hat{A} \in$ E, $\hat{A} \leq^p_m A$, and $\hat{A}$ is E_k-bi-immune.*

Proof. Obviously, $\hat{A} \leq^p_m A$ via $f(x) = 0^{|x|}$. Since $|f(x)| = |x|$ and $A \in$ E it follows that $\hat{A} \in$ E too. It remains to show that $\hat{A}$ is E_k-bi-immune. By symmetry, it suffices to show that there is no infinite set $B \in \mathrm{E}_k$ such that $B \subseteq \hat{A}$. For a contradiction assume that such a set B exists. Then, for $\tilde{B} = \{0^n : \exists\, x \in B\ (|x| = n)\}$, $\tilde{B} \in \mathrm{E}_{k+1}$, $\tilde{B}$ is infinite and $\tilde{B} \subseteq A$. So A is not E_{k+1}-bi-immune contrary to assumption. □

Theorem 4.2. *There is a tally set $A \in$ E which is strongly E-nontrivial.*

Proof. Since there are E_2-bi-immune sets in E, it follows from Lemma 4.1 that there is a length language $A_1 \in$ E which is E_1-bi-immune. By Theorem 2.6, A_1 is strongly E-nontrivial. Since, for the tally set $A = A_1 \cap \{0\}^*$, A is in E and A is p-m-equivalent to A_1, it follows from p-m-invariance of strong E-nontriviality that A has the desired properties. $\square$

In order to distinguish E-nontriviality from strong E-nontriviality we will show that there are E-nontrivial sets of very low density whereas strongly E-nontrivial sets in E do not have (sufficiently easily recognizable) exponential gaps. These observations will allow us to argue that there are exptally sets in E which are E-nontrivial whereas no such set is strongly E-nontrivial.

In order to prove the existence of E-nontrivial sets of very low density in E, we observe that *any* sufficiently complex tally set in E is E-nontrivial.

Theorem 4.3. *Let $A \in \mathrm{E} \setminus \mathrm{E}_1$ be tally. Then A is* E*-nontrivial.*

PROOF. Given $k \geq 1$, it suffices to give a set B_k such that $B_k \leq^p_m A$ and $B_k \in \mathrm{E} \setminus \mathrm{E}_k$. For $n \geq 0$ let

$$y_n = 0^{\lfloor \frac{n}{k+1} \rfloor} 1 z_n$$

where z_n in the nth binary string with respect to the canonical ordering, and let

$$B_k = \{y_n : n \geq 0 \ \&\ 0^n \in A\}.$$

In order to show that B_k has the required properties, first note that the strings y_n have the following properties.

(i) For almost all n, $\frac{n}{k+1} \leq |y_n| \leq \frac{n}{k}$,

(ii) for a given number n, y_n can be computed in $poly(n)$ steps, and

(iii) for a given string x, in $poly(|x|)$ steps we can tell whether $x = y_n$ for some n and if so compute the unary representation 0^n of the unique n with this property.

Now, since $y_n \in B_k$ if and only if $0^n \in A$, (iii) implies that $B_k \leq^p_m A$ via f where $f(x) = 0^n$ if $x = y_n$ and $f(x) = 1 \notin A$ if x is not among the strings y_n, $n \geq 0$. Moreover, by the first inequality in (i), $|f(x)| \leq (k+1) \cdot |x|$ for almost all x whence, by $A \in \mathrm{E}$, $B_k \in \mathrm{E}$ too. Finally, $B_k \notin \mathrm{E}_k$. Namely, otherwise, for sufficiently large n, $A(0^n)$ can be computed in $O(2^n)$ steps (contrary to $A \notin \mathrm{E}_1$), by first computing y_n (which, by (ii), can be done in $poly(n)$ steps) and then computing $B_k(y_n)$ (which, by assumption and by the second inequality in (i), can be done in $O(2^{k \cdot |y_n|}) \leq O(2^n)$ steps). $\square$

Corollary 4.4. *Let B be an infinite tally set such that $B \in \mathrm{E}$. There is an* E*-nontrivial set A in* E *such that $A \subseteq B$.*

Corollary 4.4 is a direct consequence of Theorem 4.3 and the following observation.

Lemma 4.5. *Let B be any infinite set in* E *and let $k \geq 1$. There is a subset A of B such that $A \in \mathrm{E} \setminus \mathrm{E}_k$.*

Proof. This can be shown by a straightforward diagonalization. Alternatively, we can use the fact that, for any $k' \geq 1$, there is an $\mathrm{E}_{k'}$-bi-immune sets in E. Namely, given $k' \geq k$ such that $B \in \mathrm{E}_{k'}$, let $A = B \cap C$ where C is any $\mathrm{E}_{k'}$-bi-immune set in E. Then, obviously, $A \subseteq B$ and, by closure of E under intersection, $A \in \mathrm{E}$. Finally, by $\mathrm{E}_{k'}$-bi-immunity of C and by choice of B, $A = B \cap C \notin \mathrm{E}_{k'}$. So, by $k' \geq k$, $A \notin \mathrm{E}_k$. □

Now, Corollary 4.4 directly implies that there are exptally E-nontrivial sets in E. Here a set A is *exptally* if $A \subseteq \{0^{\delta(n)} : n \geq 0\}$ where $\delta : \mathbb{N} \to \mathbb{N}$ is the *iterated exponential function* inductively defined by $\delta(0) = 0$ and $\delta(n+1) = 2^{\delta(n)}$. (Intuitively, an exptally set may be viewed as the unary encoding of a tally set or as the unary encoding of the unary encoding of an arbitrary set.)

Corollary 4.6. *There is an* E*-nontrivial set* A *in* E *which is exptally.*

Proof. Since $\{0^{\delta(n)} : n \geq 0\} \in \mathrm{P}$, this is immediate by Corollary 4.4. □

Remarks 4.7. *1. It might be of interest to note that Theorem 4.3 and Corollary 4.4 in general fail for sparse sets in place of tally sets. A counter example provides the* E*-trivial set* A *constructed in the proof of Theorem 3.4 which is sparse (in fact contains exactly one string of each length). Moreover,* A *satisfies the condition*

$$\forall^\infty x \; (|f(x)| \leq 2 \cdot |x| \;\; or \;\; f(x) \notin A)$$

for all polynomial-time computable functions f*. Since this property is inherited by any subset of* A*, it follows by the Boundedness Lemma (Lemma 3.5) that all subsets of* A *in the classe* E *are* E*-trivial too.*

2. Similarly, in Theorem 4.3 and Corollary 4.4 the assumptions that $A \in E$ *and* $B \in \mathrm{E}$*, respectively are necessary. Namely, there is a tally set* $A \notin \mathrm{E}_1$ *such that*

$$\forall \, B \; (B \subseteq A \Rightarrow B \text{ E-}trivial).$$

Such a set A *can be obtained a diagonal argument. It suffices to ensure that for any set* $E \in \mathrm{E}$ *and for any polynomial time computable function* f *such that* $f(E)$ *is infinite,* $f(E) \not\subseteq A$*.*

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We first observe that no exptally set in E is strongly E-nontrivial.

Theorem 4.8. *Let* $A \in \mathrm{E}$ *be exptally. Then* A *is not strongly* E*-nontrivial.*

Since any strongly E-nontrivial set has an E_1-bi-immune (hence P-bi-immune) predecessor, it suffices to show the following.

Lemma 4.9. *Let* A *and* B *be sets such that* $A \in \mathrm{E}$*,* A *is exptally, and* $B \leq^p_m A$*. Then* B *is not* P*-bi-immune.*

The idea of the proof of Lemma 4.9 is as follows. Given a polynomial bound p for a p-m-reduction f from B to A, let $D = \{0^{\delta'(n)} : n \geq 0\}$ where $\delta'(n)$ is the least number m such that $p(m+1) \geq \delta(n)$. Then D is infinite and polynomial-time computable. Moreover, for $x \in D$, $B(x)$ can be computed in polynomial time using the reduction $B(x) = A(f(x))$. Namely if, for $x = 0^{\delta'(n)}$, $f(x) = 0^{\delta(r)}$ then $r < n$ whence, by definition of δ and δ' and by $A \in \mathrm{E}$, $A(f(x))$ can be computed in $poly(|x|)$ steps; and if $f(x)$ is not of the form $0^{\delta(r)}$, then $B(x) = A(f(x)) = 0$.

The observation that exptally sets in E cannot be strongly E-nontrivial can be generalized as follows.

Theorem 4.10. *Let A and B be sets in* E *such that $B \subseteq \{0\}^*$, B is infinite, and*

$$\forall\, n\, (0^n \in B \;\Rightarrow\; A \cap \{x : n < |x| < 2^n\} = \emptyset) \tag{11}$$

holds. Then A is not strongly E*-nontrivial.*

PROOF. By $A, B \in \mathrm{E}$ fix $k \geq 1$ such that $A, B \in \mathrm{E}_k$. It suffices to show that there is no E_k-bi-immune set (in E) which can be p-m-reduced to A. So, given any $D \in \mathrm{P}_m(A)$, in order to show that D is not E_k-bi-immune fix f such that $D \leq^p_m A$ via f and, by polynomial time computability of f, fix n_0 such that $|f(x)| < 2^{|x|}$ for all strings x of length $> n_0$. Since the tally set B is infinite and in E_k, it suffices to show that, for $n > n_0$ such that $0^n \in B$, $D(0^n)$ can be computed in $O(2^{kn})$ steps. Since f is polynomial time computable and since $D(0^n) = A(f(0^n))$, for n such that $|f(0^n)| \leq n$, this follows from $A \in \mathrm{E}_k$, while, for n such that $|f(0^n)| > n$, $A(f(0^n)) = 0$ by (11) and $n > n_0$. □

Theorem 4.10 implies that many constructions (of sets in E) in the theory of the polynomial-time degrees which are based on so-called gap languages (see e.g. Section 3 of [3]) yield sets which are not strongly E-nontrivial.

4.1. Exptally Sets and Nontriviality

In contrast to the above negative result on strong E-nontriviality for exptally sets, there are exptally sets in E which are E-nontrivial. In fact, *any* sufficiently complex exptally set $A \in \mathrm{E}$ is E-nontrivial.

Theorem 4.11. *Let $A \in \mathrm{E} \setminus \mathrm{E}_1$ be exptally. Then A is* E*-nontrivial.*

PROOF (IDEA). Given $k \geq 1$, we have to show that there is a set $A_k \leq^p_m A$ such that $A_k \in \mathrm{E} \setminus \mathrm{E}_k$. Such a set A_k is obtained from A by compressing strings of the form $0^{\delta(n)}$ by the factor k: $A_k = \{0^{\lfloor \frac{\delta(n)}{k} \rfloor} : 0^{\delta(n)} \in A\}$. Note that, by A being exptally, all (but a finite amount) information about A is coded into A_k in k-compressed form. This easily implies the claim. □

Note that, by a straightforward diagonalization, there is an exptally set $A \in \mathrm{E} \setminus \mathrm{E}_1$. So, by Theorems 4.11 and 4.8, there is an E-nontrivial set in E which is not strongly E-nontrivial.

4.2. The Hierarchy of Weak Completeness Notions

By the above given differences in the possible densities of the sets with the various weak completeness properties we immediately get the following hierarchy theorem.

Theorem 4.12. *For any set A the following hold.*

$$\begin{array}{c} A \text{ E-}\textit{hard} \\ \Downarrow \\ A \text{ E-}\textit{measure hard} \\ \Downarrow \\ A \text{ E-}\textit{category hard} \\ \Downarrow \\ A \textit{ strongly } \text{E-}\textit{nontrivial} \\ \Downarrow \\ A \text{ E-}\textit{nontrivial} \\ \Downarrow \\ A \textit{ intractable} \end{array} \tag{12}$$

Moreover all implications are strict and witness sets A for strictness can be found in E.

5. On the Information Content of E-Nontrivial and Strongly E-Nontrivial Sets

In the preceding section we have distinguished E-nontriviality from the stronger weak completeness notions for E by analysing the possible densities of sets with these properties. Here we present another difference in the sense of information content. We look at the following question: If we split a (weakly) complete set A into two parts A_0 and A_1, is at least one of the parts (weakly) complete again? As we will show, for E-nontriviality the answer is YES whereas for the other weak completeness notions the answer is NO.

In order to make our question more precise we need the following notion. A splitting of a set A into two disjoint sets A_0 and A_1 is a *p-splitting* if there is a set $B \in \mathrm{P}$ such that $A_0 = A \cap B$ and $A_1 = A \cap \overline{B}$. Note that for a p-splitting (A_0, A_1) of A, $A_0, A_1 \leq^p_m A$ and $A =^p_m A_0 \oplus A_1$ (where $A_0 \oplus A_1$ is the effective disjoint union $\{0x : x \in A_0\} \cup \{1y : y \in A_1\}$ of A_0 and A_1). So, intuitively, a p-splitting decomposes a problem A into two subproblems A_0 and A_1 so that for solving A it suffices to solve both A_0 and A_1.

Now Ladner [17] has shown that any computable intractable set A can be p-split into two lesser intractable problems, i.e., into problems $A_0, A_1 \notin \mathrm{P}$ such that $A_0, A_1 <^p_m A$. So, in particular, any E-complete set A can be p-split into two incomplete sets. In fact, by analysing Ladner's proof, the set $B \in \mathrm{P}$ defining the p-splitting is a gap language. So, by Theorem 4.10, we obtain the following stronger observation from Ladner's proof.

Lemma 5.1. *Let $A \in \mathrm{E} \setminus \mathrm{P}$. There is a p-splitting of A into sets $A_0, A_1 \notin \mathrm{P}$ such that A_0 and A_1 are weakly* E*-trivial.*

So, in particular, any E-complete (E-measure complete, E-category complete, strongly E-nontrivial) set A has a p-splitting into sets A_0 and A_1 which are not E-complete (E-measure complete, E-category complete, strongly E-nontrivial). For E-nontriviality, however, the corresponding fact fails.

Theorem 5.2. *Let A be* E*-nontrivial and let (A_0, A_1) be a p-splitting of A. Then A_0 is* E*-nontrivial or A_1 is* E*-nontrivial (or both).*

The key to the proof is the simple observation that, for a p-splitting C_0, C_1 of a set $C \notin \mathrm{E}_k$, $C_0 \notin \mathrm{E}_k$ or $C_1 \notin \mathrm{E}_k$.

Proof (Sketch). For a contradiction assume that A_0 and A_1 are E-trivial. Fix k_i such that $\mathrm{P}_m(A_i) \cap \mathrm{E} \subseteq \mathrm{E}_{k_i}$ $(i = 0, 1)$ and let $k = \max(k_0, k_1)$. Moreover, fix $B \in \mathrm{P}$ such that $A_0 = A \cap B$ and $A_1 = A \cap \overline{B}$. Finally, by E-nontriviality of A, fix a set $C \in \mathrm{E} \setminus \mathrm{E}_k$ such that $C \leq_m^p A$ and let f be a polynomial-time computable function such that $C \leq_m^p A$ via f. Then, for $D = \{x : f(x) \in B\}$, $D \in \mathrm{P}$. Now, consider the p-splitting $C_0 = C \cap D$ and $C_1 = C \cap \overline{D}$ of C given by D. Then $C_0, C_1 \in \mathrm{E}$ and, as one can easily check, $C_0 \leq_m^p A_0$ and $C_1 \leq_m^p A_1$. Since, by the above observation, $C_0 \notin \mathrm{E}_k$ or $C_1 \notin \mathrm{E}_k$, this contradicts the choice of k. $\square$

For an E-complete set A we obtain the following interesting variant of Theorem 5.2, which says that for a proper splitting of an E-complete set *both* parts are E-nontrivial.

Theorem 5.3. *Let A be* E*-complete and let (A_0, A_1) be a p-splitting of A such that $A_0, A_1 <_m^p A$. Then A_0* and *A_1 are* E*-nontrivial.*

We omit the quite involved proof which requires some new result on the distribution of the E_k-bi-immune sets in E.

6. Further Results and Open Problems

We conclude with a short summary of some other results on our new weak completeness notions which will appear somewhere else.

6.1. Comparing Weak Hardness for E *and* EXP

While, by a simple padding argument, E-hardness and EXP-hardness coincide, Juedes and Lutz [16] have shown that E-measure hardness implies EXP-measure hardness whereas the converse in general fails. Moreover, by using similar ideas, the corresponding results have been obtained for category hardness (see [4]).

Now we can easily adapt the concepts of (strong) nontriviality for E to the polynomial-exponential time class EXP by replacing E and E_k in the definitions by EXP and EXP_k, respectively. Then, the arguments of [16] can be easily duplicated to show that strong E-nontriviality implies strong EXP-nontriviality but in general not vice versa.

For clarifying the relations between E-nontriviality and EXP-nontriviality, however, the above arguments fail and new much more sophisticated techniques have to be employed. As it turns out, in contrast to the above results, E-nontriviality and EXP-nontriviality are independent. I.e. neither E-nontriviality implies EXP-nontriviality nor EXP-nontriviality implies E-nontriviality. For details see [5].

6.2. Weak Hardness Under Weak Reducibilities

The classical approach to generalize hardness notions is to generalize (weaken) the reducibilities underlying the hardness concepts, i.e., to allow more flexible codings in the reductions. So Watanabe [22] has shown that weaker reducibilities than p-m-reducibility like p-btt-reducibility (bounded truth-table), p-tt-reducibility (truth-table) and p-T-reducibility (Turing) yield more E-complete sets. For measure-completeness and category-completeness similar results have been shown in [8] (for both E and EXP). For strong nontriviality we obtain the corresponding results, i.e., a complete separation of p-m, p-btt, p-tt, and p-T (for both E and EXP), by fairly standard methods. For nontriviality, however, the separations of E-nontriviality under p-m, p-btt, p-tt, and p-T reducibilities require some quite involved and novel speed-up diagonalization technique. The reason why standard methods fail in this setting might be explained by the fact that - in contrast to the above results - EXP-nontriviality under p-m, p-btt, and p-tt coincide. See [6] for details.

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