Summery lechers "Limits theorems"for students of 4 course speciality of mathematics.

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Lecthes 1–2.

Introduction. Probability theory with mathematical statistics adjoining it, fairly are among the mathematical disciplines having the most close connections with practice. To number of advantages of this disciplines it should be noted that methods offered them are capable to work and deliver qualitative and quantative information in extremely adverseconditions when about the studied phenomenon, it is known of factors generating it and the mechanism of its formation very little or even it it ist't know. This feture often does probability theory pervoprokhodchik in science. Moreover, when nature of studied regularities, behind the theory probability is found essentionally stochastic (i.e. difined of case) the learning role of the supplier of mathematical models and methods of the quantative analysis remains. Interesting to mark that first from such methods appeared as early as the XVI century, till the basic ideas of theory probability were formed. This method born during practical researches of the casual phenomena was taken to the supervision of frequencies of appearence of unforeseeable in the long row of the tests repeated unconnected inter se. Found on a big and various matherial a phenomenon of stabilization of frequancies of emergence of a casual event first had no jastification and it was perceived as purely empirical fact. Therefore emergence of the theoretical constractions explaining at the level of mathematical models this phenomenon, coused a great interest both from mathimaticans, and from experts. The well-known theorem published in 1713 of Ia.Bernoulli called subsequantly by the law of large nambers, became this remarkable result, which has laid the foundation of probability theory as siences. The first Bernoulli who has followed the theorem result – Moivre-Laplass so-called theorem represented specification of the theorem Bernoulli. Presently already strongly there was to divide a tradition limit theorems into two, as through the independent groups uniting results like the law of the big large numbers (Bernulli's result is first of which) and results like the central limit theorem (which elementary representative Moivre-Laplace‘s theorem is). Accourding to same experts, basic distinction of theorem of these two groups isn‘t present, and this devision should be considered simply as a tribute convergence of classification. The similar perception of these two groups of theorems can be challenger, hawever the main thing is nevertheless their obgectivity. At formal creation of a course of probability theory limit theorems appear in the form of same kind of superstructure over elementary heads of probability theory in whom all tasks have final, purely arifmetic character. Actually however, the informative value of probability theory reveals only limit theorems. Moreover, without limit theorems there can't be undestoodly real content of the most inital consept-consept of probability. Really, all informative value of probability theory is consed by that the mass casual phenomena in the cumulative action create strict, not casual regularities; the concept of mathematical probability would be fruitless if it didn't find the implementation in the form of the frequency of emergence of any result at mass repetition of uniform conditions (at unlimited increase in number of tests, as mush as exact and reliable). Therefore the elementary arithmetic culculations of probabilitions relating to gamblings, in works of mathematicians before Ja.Bernoulli's work, it is possible to consider as a probability theory prehistory, and its real history begins with and its real history begins with Bernoulli's theorems (1713) and Moivre (1730). To these limit theorems, as the main achivements of probability theory to P.L.Chebysheva, it is necessery to add Poisson's three more theorems from which one generalizes Bernoulli's theorem, another Moivre-Laplace's theorem and the third leads to Poisson's distribution. For clear undestanding further, it is useful to provide here a little upgraded formulations of the listed theorems.

The first four of them treat sequence of *indepandant tests*

$$U^1, U^2, U^3, \dots$$

in each of which there can be two outcomes У (success) and Н (failure). Test we will designate probabilities of these events throught

$$p_j = P(L^j), \quad q_j = (1 - p_j),$$

and from among the first tests we will designate number of actually appeared progress throught μ_n.

In the first two theorems so-called uniform tests, in which all are considered p_j are equal to the same number $p \quad (0 < p < 1)$.

1) **Bernouli's theorem.** At any $\varepsilon > 0$

$$P\left(\left|\frac{\mu_n}{n} - p\right| > \varepsilon\right) \to 0,$$

when $n \to \infty$.

2) **Theorem Moivre—Laplace.**

$$P\left\{a \leqslant \frac{\mu_n - np}{\sqrt{npq}} \leqslant b\right\} \to \frac{1}{\sqrt{2\pi}} \int\limits_a^b e^{-\frac{x^2}{2}} dx,$$

and uniformly a and b, when $n \to \infty$.

In the following two theorems of probability p_n may depend on n, but subject to the condition, series

$$\sum_{n=1}^{\infty} p_n q_n$$

deverges. In these formulas, the notations

$$p_1 + p_2 + \dots + p_n = A_n,$$

$$p_1 q_1 + p_2 q_2 + \dots + p_n q_n = B_n^2.$$

3) **The law of large numbers in the form Poisson.** At any $\varepsilon > 0$

$$P\left(\left|\frac{\mu_n}{n} - \frac{A_n}{n}\right| > \varepsilon\right) \to 0,$$

when $n \to \infty$.

4) **The main limit theorem in the form Poisson.**

$$P\left\{a \leqslant \frac{\mu_n - A_n}{B_n} \leqslant b\right\} \to \frac{1}{\sqrt{2\pi}} \int\limits_a^b e^{-\frac{x^2}{2}} dx,$$

evenly relative a and b, when $n \to \infty$.

The fifth of theorems interesting us treats sequence of series (the scheme of series)

$$\begin{array}{l}
U^{11}, \\
U^{21},\ U^{22} \\
U^{31},\ U^{32},\ U^{33}, \\
\dots\dots\dots\dots\dots\dots\dots \\
U^{n1},\ U^{n2},\ U^{n3}, \dots,\ U^{nn}, \\
\dots\dots\dots\dots\dots\dots\dots
\end{array}$$

in which tests of one series are mutually independent among themselves with distribution of probabilities $p_n + q_n = 1$, in which tests of one series are mutually independant among themselves with distribution of probabilities depending only from series member of actually appeared number. Throught μ_n we will designate number of actuallyappeared achievements in n - a series.

5) **Poisson's limit theorem for rare events.** If

$$np_n \to \lambda$$

at $n \to \infty$, that

$$P(\mu_n = m) \to \frac{\lambda^m}{m!} e^{-\lambda}.$$

We will notice that the provided scheme of series looks a little artifical. Really we a always at least mentally, can complite missing series. However, this scheme, explains dependance of probabilities of outcomes in each series from n. Therefore, the last formula can be applied and at usual consecutive tests, at rather small p and moderate np. In this regard there is question. At the fixed and rather big n, for binomial distribution to define at what values p intervals (0,1), that is the smaller rest, is the best one of the given approximations normal or Poisson. In other words, to what values of parameter of p intervals (0,1) it is better to apply Moivre-Laplace's theorem, and for Poisson's theorem. Before formulating the corresponding theorem, we will consider necessary designations. We will put:

$$B(m) = \frac{n!}{m!\,(n-m)!} p^m q^{n-m},\ \lambda = np;$$

$$\Pi_1(m),\ m < 0; \quad \Pi_1 = \frac{(np)^m}{m!} e^{-(np)},\ m \geqslant 0;$$

$$\Pi_2(m) = \frac{1}{\sqrt{2\pi npq}} e^{-\frac{x^2}{2}},\ x = \frac{m - np}{\sqrt{npq}};$$

$$\Pi_3(m) = 0,\ m < 0; \quad \Pi_3(m) = \frac{(nq)^m}{m!} e^{-nq},\ m \geqslant 0;$$

$$\rho_k(p,n) = \sum |B(m) - \Pi_k(m)| \ \ (k = 1, 2, 3).$$

Theorem (Yu.V.Prokhorov). At $n \to \infty$

$$\rho_1(p,n) = c_1 p + p\,O\left(\min\left(1,(np)^{-\frac{1}{2}}\right)\right), \quad c_1 = \sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}} = 0,483\ldots$$

$$\rho_2(p,n) = c_2\frac{|q-p|}{\sqrt{npq}} + O\left(\frac{1}{\sqrt{npq}}\right), \quad c_2 = \frac{1}{3\sqrt{2\pi}}\left(1+4e^{-\frac{3}{2}}\right) = 0,251\ldots$$

Consequence. There is such number $c_3 = 0,637\ldots$, that

$$\min \rho_k(p,n) = \begin{cases} \rho_1(p,n) & p < c_3 n^{-\frac{1}{3}} + O\left(n^{-\frac{2}{3}}\right), \\ \rho_2(p,n) & c_3 n^{-\frac{1}{3}} + O\left(n^{-\frac{2}{3}}\right) \leqslant p < 1 - c_3 n^{-\frac{1}{3}} + O\left(n^{-\frac{2}{3}}\right), \\ \rho_3(p,n) & p \geqslant 1 - p n^{-\frac{1}{3}} + O\left(n^{-\frac{2}{3}}\right). \end{cases}$$

Iacob Bernoulli's contemporaries and the subsequent generations of scientists saw the big practical importance of the law of large numbers that it was the peculiar bridge which has connected the theory and practice. With rare expection the probability theory has no opportunity to determine by purely speculative way knowledge of probabilities or the related sizes serving as input parameters of considered mathematical model.This knowledge should be got by carrying out a series of corresponding experiments being guided by indications of the law of large numbers.

In process of distribution of action of the law of large numbers on model of the law of large numbers on model of an escalating community the sphere of its mathematical community of its application extended also. However, passing various stages of generalization, the law of large numbers always remained thus the fact purely mathematical, only to a greater or lesser extent reflecting objective regularities of the real world. Therefore about a prototype of the mathematical law of large numbers it is possible to speak as about some internal property of many real processes representing very widespread phenomenon. Having, apparently, desire to give to definition of the law of large numbers probably bigger coverage A.N.Kolmogorov as follows formulated his essence in the relevant article of the big Soviet encyclopedia: The law of large numbers – "the general principle owing to which cumulative action of a large number of random factors brings, under the general conditions same very, to result not depending almost from a case". Thus, the law of large numbers has as thought two treatments. One - mathematical, connected with concrete mathematical models, and the second - more the general, laving for this framework. The second treatment is connected with phenomenon of education quite often noted in practice of same extent directed action against a large number of the hidden or visible operating factor, externally such focus of not having. Examples connected with the second treatment it is possible to give a set if to address to economy (for example, a pricing phenomenon in the free market), the social sphere (formation of public opinion on this or that question), ets.

Lections 3–4.

Preliminary dates

Characteristic function of a random variable ξ is called

$$f_\xi(t) = Me^{it\xi} = \int\limits_{-\infty}^{\infty} e^{it\xi}dF(x).$$

we will note same properties for characteristic functions

1.Characteristic function is evenly continuous on all numerical straight line and meets conditions:

$$f(0) = 1, \quad |f(t)| = 1.$$

2. If $\eta = a\xi + b$,

$$f_\eta(t) = e^{ibt} f_\xi(at).$$

3. If $\xi_1, \xi_2, \ldots, \xi_n$ independent random variables,

$$f_{\xi_1 + \cdots + \xi_n}(t) = f_{\xi_1}(t) \ldots f_{\xi_n}(t)$$

4.As the moments α_n and the absolute moments β_n a random variables ξ are called respectively a size $M\xi^n$ and $M|\xi|^n \quad (n > 0)$. In term of function of distribution

$$M\xi^n = \int_{-\infty}^{\infty} x^n dF(x), \quad M|\xi|^n = \int_{-\infty}^{\infty} |x|^n dF(x).$$

If there is an absolute moment n

$$\beta_n = \int_{-\infty}^{\infty} |x|^n dF(x)$$

that are all derivatives of characteristic function including to a derivative n - are order. And

$$f^k(0) = i^k \int_{-\infty}^{\infty} x^k dF(x).$$

5. In there is an other moment $n + \delta$,

$$\beta_{n+\delta} = \int_{-\infty}^{\infty} |x|^{n+\delta} dF(x) \quad 0 < \delta \leq 1.$$

then fairly following decomposition of characteristic function in a vicinity of zero point.

$$f_\xi(t) = 1 + \frac{\alpha_1}{1!} it + \frac{\alpha_2}{2!}(it)^2 + \cdots + \frac{\alpha_n}{n!}(it)^n \frac{2^{1-\delta}\beta_{n+\delta}\theta|t|^{n+\delta}}{(1+\delta)(2+\delta)\ldots(n+\delta)}, \qquad |\theta| < 1.$$

6. Between distribution function $F(x)$ and characteristic functions there is a bunique complience:

a) $F(x)$ and $f(t)$ unambiguously define each other

$$f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x), \quad F(x) = \frac{1}{2\pi} \lim_{\{y \to -\infty} \lim_{T \to \infty} \int_{-T}^{T} \frac{e^{-ity} - e^{=itx}}{it} f(t) dt.$$

б) If the sequence of function $\{F_n(x)\}$ meets to $F(x)$ in each point of a continuity $F(x)$, then the sequence of the corresponding characteristic function $\{f_n(t)\}$ meets to characteristic function $f(t)$ evenly, in each final interval $|t| \leqslant T$.

Back, if $\{f_n(t)\} \to f(t)$, then the sequence of functions of distribution $\{F_n(t)\}$ meets generally to $F(x)$, and its necessary $f(t)$ there is a characteristic function of limit function $F(x)$.

For reduction of records further, we will adhere to the following designations for law distributions.

1) Normal distribution — $N(a, \sigma)$;
2) Bernulli's distribution — $B(n, p)$;
3) Poisson's distribution — $\Pi(\lambda)$;
4) Indicate distribution — $\pi(x)$;
5) Uniform distribution — $r(x)$;
6) Degenerate distribution — $R(0)$.

7. We will consider characteristic function of the most important distribution. 1) Degenerate distribution

$$P\{\xi = 0\} = 1, \qquad f(t) = 1.$$

2) Normal distribution

$$\varphi(x) = \frac{1}{2\pi} e^{-\frac{x^2}{2}} \qquad f(t) = e^{-\frac{t^2}{2}}.$$

3) Poisson's distribution

$$P\{\xi = m\} = \frac{\lambda^k}{k!} e^{-\lambda}, \qquad f(t) = e^{\lambda(e^{it}-1)}.$$

4) Bernulli's distribution

$$P\{\xi_n = m\} = C_n^m p^m q^{n-m}, \qquad f(t) = \left(pe^{it} + q\right)^n.$$

5) Indicate distribution

$$p(x) = ae^{-ax} \quad (x \geqslant 0), \qquad f(t) = \frac{a}{a - it} \quad (a > 0).$$

6) Uniform distribution

$$p(x) = \frac{1}{2l}, \quad |x| \leqslant l, \qquad f(t) = \frac{\sin lt}{lt}.$$

7) Let ξ_1 and ξ_2 – independent random variables with distribution function $F_1(x)$ and $F_2(x)$ accordingly. Distribution $F(x)$ of summer $\xi_1 + \xi_2$ is:

$$F(x) = \int\limits_{-\infty}^{\infty} F_1(x-y) dF_2(y) = \int\limits_{\infty}^{\infty} F_2(x-y) f F_1(y).$$

Over distribution functions putting in compliance to two functions to distributions $F_1(x)$ and $F_2(x)$ function $F(x)$ on the specified formula we will call operation composition or convolution of functions distribution designation

$$F(x) = F_1(x) * F_2(x).$$

equally distributed sizes. We will write also $F^{*n}(x)$ for convolution designation n equally distributed sizes. Operation of composition is commutative and associative operation. We will note still that least one of a component $F_1(x), F_2(x)$ it is absolutely continuous, that will be and $F(x)$.

Lectures 5–6.

First limit theorems and limit laws.

We will consider Bernoulli's scheme with probability of success in each separate test $p > 0$. We will designate over μ_n number success in n independent tests.

$$\mu_n = \xi_1 + \xi_2 + \cdots + \xi_n.$$

when

$$\xi_k \in \{0, 1\}, \quad p(\xi = 1) = p, \quad p(\xi = 0) = q, \quad (p + q = 1).$$

Theorem (Bernoulli)

For any $\varepsilon > 0$

$$\lim_{n\to\infty} P\left\{|\frac{\mu_n}{n} - p| > \varepsilon\right\} = 0.$$

or that is equivalent, by $n \to \infty$

$$R\left(\frac{\mu_n - np}{n}\right) \to R(0)$$

Proof. We will show at first equivalence of these expressions. Function of distribution of the generate random variable has an appearance:

$$E(x) = \begin{cases} 1, & \text{если } x > 0, \\ 0, & \text{если } x \leqslant 0. \end{cases}$$

$$F_n(x) = P\left\{\frac{\mu_n - np}{n} \leqslant x\right\}.$$

Let ε any positive number.

$$P\left\{\left|\frac{\mu_n - np}{n}\right| \leqslant \varepsilon\right\} = P\left\{-\varepsilon < \frac{\mu_n - np}{n} < \varepsilon\right\} = F_n(\varepsilon) - F_n(-\varepsilon).$$

According to statement it aspires to unit, i.e.

$$F_n(\varepsilon) - F_n(-\varepsilon) \to 1 \quad (\varepsilon > 0)$$

It means that

$$F_n(x) \to \begin{cases} 1, & \text{если } x > 0, \\ 0, & \text{если } x \leqslant 0. \end{cases}$$

The return is also right, i.e. if

$$R\left(\frac{\mu_n - np}{n}\right) \to R(0),$$

then

$$\lim_{n\to\infty} P\left\{|\frac{\mu_n}{n} - p| > \varepsilon\right\} = 0.$$

For the proof of the approval of the theorem it is enough to show that $R((\mu_n - np)/n)$ meets to characteristic function of the degenerate law. We will notice that characteristic function for $E(x)$ it is equal 1.

Let $R((\mu_n - np)/n)$ has the characteristic function $f_n(t)$, then

$$f_n(t) = Me^{it\sum\limits_{k=1}^{n}\frac{\xi+k-p}{n}} = \prod_{k=1}^{n} Me^{it\frac{\xi_k - p}{n}} = \left(pe^{\frac{itq}{n}} + qe^{-\frac{itp}{n}}\right)^n.$$

Now we will use decomposition in a row Makloren

$$f_n(t) = \left[p\left(1 + \frac{itq}{n} + o\left(\frac{t}{n}\right)\right) + q\left(1 - \frac{itp}{n} + o\left(\frac{t}{n}\right)\right)\right]^n = \left[1 + o\left(\frac{t}{n}\right)\right]^n \to 1.$$

As was to be shown.

Theorem (Moivre — Laplace)

We will prove that at $n \to \infty$,

$$F_n(x) = P\left\{\frac{\mu_n - np}{npq} \leqslant x\right\} \to \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-\frac{y^2}{2}} dy = \Phi(x)$$

$$R\left(\frac{\mu_n - np}{\sqrt{npq}}\right) \to N(0, 1).$$

Proof. On to show convergence of characteristic function enough

$$f_n(t) \to e^{-\frac{t^2}{2}}.$$

$$f_n(t) = Me^{it\sum\limits_{k=1}^{n}\frac{\xi+k-p}{n}} = \prod_{k=1}^{n} Me^{it\frac{\xi_k - p}{n}} = \left(pe^{\frac{itq}{n}} + qe^{-\frac{itp}{n}}\right)^n.$$

$$Me^{it\frac{\xi_k - p}{\sqrt{npq}}} = e^{-\frac{itp}{\sqrt{npq}}} Me^{\frac{it\xi_k}{\sqrt{npq}}} = e^{-it\sqrt{\frac{p}{nq}}}\left(pe^{\frac{it}{\sqrt{npq}}} + q\right) =$$

$$= pe^{it\frac{1-p}{\sqrt{npq}}} + qe^{-it\sqrt{\frac{q}{np}}} = pe^{it\frac{q}{\sqrt{npq}}} + qe^{-it\sqrt{\frac{q}{np}}}$$

$$f_n(t) = \left[p\left(1 + it\sqrt{\frac{q}{np}} + \frac{(it)^2 q}{2!np} + o\left(\frac{t^2}{n}\right)\right) + q\left(1 - it\sqrt{\frac{p}{nq}} + \frac{(it)^2 p}{2!nq} + o\left(\frac{t^2}{n}\right)\right)\right] =$$

$$= \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n \sim \left[\left(1 - \frac{t^2}{2n}\right)^{-\frac{2n}{t^2}}\right]^{-\frac{t^2}{2}} \to e^{-\frac{t^2}{2}}.$$

Theorem (Poisson)

In his theorem Poisson altered Bernoulli scheme, suggesting that the probability $p = p_n$ depends on the total number of tests n so that $np_n \to \lambda > 0$. So, now writing ξ_{nk} and μ_{nn} ξ_k и μ_n, we get the Poisson scheme, which corresponds to the sequence of sum

$$\mu_{nn} = \sum_{k=1}^{n} \xi_{nk} \quad (n = 1, 2, \dots).$$

Theorem If $\lim\limits_{n\to\infty} np_n = \lambda > 0$, then

$$P(\mu_{nn} = k) = C_n^k p_n^k q_n^{n-k} \to \frac{\lambda^k}{k!} e^{-\lambda} \qquad (k = 0, 1, 2 \dots).$$

Proof. It suffices to show the convergence of the corresponding characteristic function.

$$f_{nn}(t) = Me^{it\sum\limits_{k=1}^{n}\xi_{nk}} = \prod_{k=1}^{n} Me^{it\xi_{nk}} = \left(p_n e^{it} + q_n\right)^n =$$

$$= \left(\frac{\lambda}{n}e^{it} + 1 - \frac{\lambda}{n} + o\left(\frac{1}{n}\right)\right)^n = \left[1 + \frac{\lambda}{n}\left(e^{it} - 1\right) + o\left(\frac{1}{n}\right)\right]^n =$$

$$= \left\{\left[1 + \frac{\lambda}{n}(e^{it} - 1) + o\left(\frac{1}{n}\right)\right]^{\frac{n}{\lambda(e^{it}-1)}}\right\}^{\lambda(e^{it}-1)} \to \exp\left\{\lambda(e^{it} - 1)\right\}$$

Lectures 7–8
Case of equally distributed composed. Theorem (Hinchin) If random variables $\xi_1, \xi_2, \dots$ independent, equally distributed also have mean, $a = M\xi_k$, at $n \to \infty$,

$$P\left\{\left|\frac{\xi_1 + \xi_2 + \dots + \xi_n}{n} - a\right| > \varepsilon\right\} \to 0.$$

$$R\left(\frac{\xi_1 + \xi_2 + \dots + \xi_n}{n} - a\right) \to R(0).$$

Proof. We will designate throught $f(t)$ characteristic function of the aligned random variable $f(t) = Me^{it(\xi_k - a)}$.

$$f_n(t) = Me^{it\sum\limits_{k=1}^{n}\frac{\xi_k - a}{n}} = \prod_{k=1}^{n} Me^{it\frac{(xi_k - a)}{n}} = f^n\left(\frac{t}{n}\right) = \left[1 + \frac{\alpha_1}{1!}\frac{it}{n} + \frac{\alpha_2}{2!}\left(\frac{it}{n}\right)^2 + \dots\right]^n$$

As $\alpha_=0 \quad M(\xi_k - a) = 0$,

$$f^n\left(\frac{t}{n}\right) = \left[1 + \frac{\alpha_2}{2!}\left(\frac{it}{n}\right) + \dots\right]^n = \left[1 + o\left(\frac{t}{n}\right)\right]^n \to 1.$$

Theorem (Hinchin — Levi)
Let $\xi_1, \xi_2, \ldots, \xi_n$ suqunce of the independant equally distributed random variables with a population mean a and dispersion σ^2. And let $S_n = \xi_1 + \cdots + \xi_n$, then at $n \to \infty$,

$$P\left\{\frac{S_n - na}{\sigma\sqrt{n}} \leqslant x\right\} \to \Phi(x)$$

Proof. We will consider a random variable $\eta_k = \frac{\xi_k - a}{\sigma}$. It is clear, that $M\eta_k = 0, \quad D\eta_k = M\eta_k^2 = 1$. Then $f\left(\frac{t}{\sqrt{n}}\right)$ – there is a characteristic function for $\frac{\eta_k}{\sqrt{n}} = \frac{\xi_k - a}{\sigma\sqrt{n}}$. Further throught $f_n(t)$ we will designate characteristic function for the rated sum.

$$\frac{S_n - na}{\sigma\sqrt{n}} = \frac{\xi_1 + \xi_2 + \cdots + \xi_n - a}{\sigma\sqrt{n}}.$$

It is enought to prove convergence of the corresponding characteristic function $n \to \infty$,

$$f_n(t) \to e^{-\frac{t^2}{2}}.$$

Really, at $n \to \infty$

$$f_n(t) = f^n\left(\frac{t}{\sqrt{n}}\right) = \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n \to e^{-\frac{t^2}{2}}.$$

The different distributed case.

Theorem (Markov) Let $\xi_1, \xi_2, \ldots, \xi_n$ sequence of independent random variables with final means $M|\xi_k|^{1+\delta}$. If

$$\lim_{n\to\infty} \frac{1}{n^{1+\delta}} \sum_{k=1}^{n} M|\xi_k|^{1+\delta} = 0,$$

then the law of large number is fair,

$$\lim_{n\to\infty} P\left\{\left|\frac{S_n}{n}\right| > \varepsilon\right\} = 0, \qquad R\left(\frac{S_n}{n}\right) \to R(0).$$

Proof. We will use decomposition of characteristic function, the theorem of a continuity and representation $\ln(1+z) = z + o(z)$, fair at $|z| < 1$.

From conditions of the theorem follows that at $n \to \infty$,

$$\max_k \frac{M|\xi_k|^{1+\delta}}{n^{1+\delta}} \leqslant \frac{1}{n^{1+\delta}} \sum_{k=1}^{n} M|\xi_k|^{1+\delta} \to 0,$$

Therefore at any fixed t,

$$f_k\left(\frac{t}{n}\right) = 1 + \frac{2^{1-\delta}\theta_{nk}|t|^{1+\delta}}{1+\delta} \frac{M|\xi_k|^{1+\delta}}{n^{1+\delta}} \to 1.$$

evenly on k. Here $f_k(t) = Me^{it\xi_k}$. We will desigant throught

$$f_n(t) = Me^{it\sum\limits_{k=1}^{n}},$$

then

$$f_n\left(\frac{t}{n}\right)=\prod_{k=1}^{n} f_k\left(\frac{t}{n}\right).$$

$$\ln f_n\left(\frac{t}{n}\right)=\sum_{k=1}^{n}\ln f_k\left(\frac{t}{n}\right)=\sum_{k=1}^{n}\ln\left[1+\left(f_k\left(\frac{t}{n}\right)-1\right)\right]=$$

$$=\sum_{k=1}^{n}\left[f_k\left(\frac{t}{n}\right)-1\right]+\sum_{k=1}^{n}\sum_{s=2}^{\infty}(-1)^s\frac{(f_k-1)^s}{s}=\sum_{k=1}^{n}\left[f_k\left(\frac{t}{n}\right)-1\right]+R_n.$$

Here we used a ratio $\ln(1+z)=z+o(|z|),\quad |z|<1$. As $f_k(t/n)\to 1$, that since some n, $\left|f\left(\frac{t}{n}\right)-1\right|<\frac{1}{2}$. Now we will be engaged in an estimate $|R_n|$.

$$|R_n|\leqslant\frac{1}{2}\sum_{k=1}^{n}\sum_{s=2}^{\infty}|f_k-1|^s=\frac{1}{2}\sum_{k=1}^{n}\frac{|f_k-1|^2}{1-|f_k-1|}\leqslant\sum_{k=1}^{n}|f_k-1|^2\leqslant$$

$$\leqslant\max_k|f_k-1|\sum_{k=1}^{n}|f_k-1|=\max_k|f_k-1|\sum_{k=1}^{n}\frac{2^{1-\delta}\theta_{nk}}{1+\delta}\frac{M|\xi_k|^{1+\delta}}{n^{1+\delta}}+o\left(\frac{c}{n^{1+\delta}}\right)\to 0$$

Thus, we have

$$f_n\left(\frac{t}{n}\right)\to 1.$$

Lectures 9–10 Theorem (Lypunov)

Let the sequence of independent random variables $\xi_1,\dots,\xi_n$ with zero population means be given. Let $S_n=\xi_1+\dots+\xi_n,\quad B_n=\sqrt{DS_n}$, if

$$\lim_{n\to\infty}\frac{1}{B_n^{2+\delta}}\sum_{k=1}^{n}M|\xi_k|^{2+\delta}=0,$$

for sum δ, then at $n\to\infty$,

$$R\left(\frac{S_n}{B_n}\right)\to N(0,1).$$

Proof. From a condition of the theorem follows

$$\max_{k\leqslant n}\left(\frac{\sigma_k}{B_n}\right)^{2+\delta}\leqslant\max_{k\leqslant n}\frac{M|\xi_k|^{2+\delta}}{B_n^{2+\delta}}\leqslant\frac{1}{B_n^{2+\delta}}\sum_{k=1}^{n}M|\xi_k|^{2+\delta}\to 0.$$

Therefore, at any fixed t and $n\to\infty$,

$$f_k\left(\frac{t}{B_n}\right)=1-\frac{t^2}{2}\frac{\sigma_k^2}{B_n^2}+\frac{2^{1-\delta}}{(1+\delta)(2+\delta)}\,\theta_{nk}\,|t|^{2+\delta}\,\frac{M|\xi_k|^{2+\delta}}{B_n^{2+\delta}}\to 1,$$

evenly on $k\leqslant n$. Therefore, at rather big n,

$$\sum_{k=1}^{n}\ln f_k\left(\frac{t}{B_n}\right)=-\frac{t^2}{2}[1+0(1)]+2\theta_n|t|^{2+\delta}\frac{1}{B_n^{2+\delta}}\sum_{k=1}^{n}M|\xi_k|^{2+\delta}\to-\frac{t^2}{2}.$$

Klassical limit theorem.

Moivre-Laplace's integrated limit theorem was a source of a big cycle of the researches having fundamental value both for the probability theory, and for appendices in naturel sciences, thecnical and economic science. To make idea of the direction of these researches, we will give to the theorem of Moivre-Laplace a bit different from. Namely, throught ξ_k let's designate number of emergence of an event A в k-m test, is equal $\sum\limits_{k=1}^{n}\xi_k$. Further,

$$M\sum_{k=1}^{n}\xi_k = np, \quad D\sum_{k=1}^{n}\xi_k = npq.$$

Therefore the theorem of Moivre-Laplace can be written down in a look: at $n\to\infty$

$$P\left\{a \leqslant \frac{\sum\limits_{k=1}^{n}(\xi_k - M\xi_k)}{\sqrt{\sum\limits_{k=1}^{n}D\xi_k}} \leqslant b\right\} \to \frac{1}{\sqrt{2\pi}}\int\limits_a^b e^{-\frac{z^2}{2}}dz,$$

Naturally there is a question: as it is intimately bound ratio (1) with an express choice of items ξ_k, whether it will take place and at weaker restrictions imposed on a cumulative distribution function of items? Statement of this task, and also its decision belongs generally to Chebyshov both its pupils Markov and Lyapunov. We will give accurate information of this condition. The reasons owing to which this results gained huge value, lie on substance of the muss phenomena which studied regularities are made by a probability theory subject. One of the major schemes on which there is use of results of probability theory in naturel sciences and technique consists in the following. Consider that process flows past under the influence of a large number of random factors, each of which has small impact on flowing past process. The researcher studying process as a whole, observer only cooperative influence of this factors. Thus, there is a problem of studying of the regularities peculiar to the sums of a large number of independent random values, each of which influences the sum a little. However, instead of studying the sums big, but a finite number of items, we will consider sequence of the sums with the increasing and large number of items and to consider that decisions are given by the limiting functions of distributions. Such transition from a terminating problem definition to the limiting is routine for the modern mathematics. So, we came to studying of the following task: the sequence independent random values is given

$$\xi_1, \xi_2, \ldots, \xi_n, \ldots$$

about which we will assume that they have terminating akspactations and dispersions. Ley's enter designations

$$a_k = M\xi_k, \quad \sigma_k^2 = D\xi_k, \quad B_n^2 = \sum_{k=1}^{n}\sigma_k^2 = D\sum_{k=1}^{n}\xi_k, \quad F_k(x) = P(\xi_k < x).$$

It is asked, what conditions it is necessary to dement from ξ_k, that a sum cumulative distribution function

$$\frac{1}{B_n}\sum_{k=1}^{n}(\xi_k - a_k) \tag{1}$$

met to the normal law?

In the following lecture we will show that performance of a condition of Lindeberg for this purpose suffices.

Lindeberg's condition and his probability sense. At any $\tau > 0$

$$\lim_{n\to\infty} \frac{1}{B_n^2} \sum_{k=1}^{n} \int\limits_{|x-a_k|>\tau B_n} (x-a_k)^2 \, dF_k(x) = 0.$$

Will find our sense of this condition.

Let's designate through A_k the event, consisting that

$$|\xi_k - a_k| > \tau B_n \qquad (k = 1, 2, \ldots, n)$$

also we will estimate probabilities

$$P\{\max_{1\leqslant k\leqslant n} |\xi_k - a_k| > \tau B_n\}$$

As

$$P\{\max_{1\leqslant k\leqslant n} |\xi_k - a_k| > \tau B_n\} = P\{A_1 + A_2 + \cdots + A_n\}$$

and

$$P\{A_1 + A_2 + \cdots + A_n\} \leqslant \sum_{k=1}^{n} P\{A_k\}.$$

that, having notice that

$$P\{A_k\} = \int\limits_{|x-a_k|>\tau B_n} (x-a_k)^2 dF_k(x) \leqslant \frac{1}{(\tau B_n)^2} \int\limits_{|x-a_k|>\tau B_n} (x-a_k)^2 dF_k(x).$$

we find an inequality

$$P\{\max_{1\leqslant k\leqslant n} |\xi_k - a_k| > \tau B_n\} \leqslant \frac{1}{\tau^2 B_n^2} \sum_{k=1}^{n} \int\limits_{|x-a_k|>\tau B_n} (x-a_k)^2 dF_k(x).$$

Owing to Lindeberg's condition, for any $\tau > 0$, the last sum at $n \to \infty$ aspires to zero. Thus, Lindeberg's condition represents a peculiar requirement of the uniform smallness $(\xi_k - a_k)/B_n$ в сумме (1). Let's note once again that the sense of conditions,sufficient for convergence of cumulative distribution functions of the sum (1) to the normal law, was found quite out by Markov and Lyapunov's researches.

Lectures 11–12

Before we will prove Lindeber's theorem, we will give some inequalities which we will use at theorem proof. It is obvious that

$$|e^{it} - 1| = \left| \int_0^t e^{ix} dx \right| \leqslant t,$$

Let's similarly receve the following inequalities

$$\begin{aligned} |e^{it}-1-it| &= \left|\int_0^t (e^{ix}-1)dx\right| \leqslant \frac{t^2}{2}, \\ \left|e^{it}-1-it+\frac{t^2}{2}\right| &= \left|\int_0^t (e^{it}-1-ix)dx\right| \leqslant \frac{t^3}{6}. \end{aligned} \tag{2}$$

Theorem 8(Lindeberg's) If sequence of mutally independent random values $\xi_1, \xi_2, \dots, \xi_n, \dots$ at any constant $\tau > 0$ meet Lindeberg's condition

$$\lim_{n\to\infty} \frac{1}{B_n^2} \sum_{k=1}^{n} \int\limits_{|x-a_k|>\tau B_n} (x-a_k)^2 \, dF_k(x) = 0, \tag{3}$$

that at $n \to \infty$ evenly designation x

$$P\left\{\frac{1}{B_n}\sum_{k=1}^{n}(\xi_k - a_k) \leqslant x\right\} \to \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-\frac{z^2}{2}} dz. \tag{4}$$

Proof. Let's enter designation

$$\xi_{nk} = \frac{\xi_k - a_k}{B_n}, \quad F_{nk}(x) = P\{\xi_{nk} < x\}.$$

It is apearent that

$$M\xi_{nk} = 0, \quad D\xi_{nk} = \frac{1}{B_n^2} D\xi_k$$

and therefore

$$\sum_{k=1}^{n} D\xi_{nk} = 1. \tag{5}$$

It is easy to be convinced that Lindaberg's condition in these designations mill assume the following air:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \int\limits_{|x|>\tau} x^2 \, dF_{nk}(x) = 0, \tag{6}$$

Caracteristic function of the sum

$$\frac{1}{B_n}\sum_{k=1}^{n}(\xi_k - a_k) = \sum_{k=1}^{n} \xi_{nk}$$

it is equal

$$\varphi_n(t) = \prod_{k=1}^{n} f_{nk}(t).$$

We need to prove, that

$$\lim_{n\to\infty} \varphi_n(t) = e^{-\frac{t^2}{2}}.$$

We will establish first of all that $f_{nk}(t)$ at $n \to \infty$ is evenly relative k aspires to 1. Really, in view of equality $M\xi_{nk} = 0$, we will be have:

$$f_{nk}(t) - 1 = \int\limits_{-\infty}^{\infty} (e^{itx} - 1 - itx) dF_{nk}(x).$$

On the basis of an inequality (1), we will be have:

$$|f_{nk}(t) - 1| \leqslant \frac{t^2}{2} \int\limits_{-\infty}^{\infty} x^2 dF_{nk}(x).$$

Let ε the arbitrariest positive number; then it is apearent, that

$$\int\limits_{-\infty}^{\infty} x^2 dF_{nk}(x) = \int\limits_{|x|\leqslant\varepsilon} x^2 dF_{nk}(x) + \int\limits_{|x|>\varepsilon} x^2 dF_{nk}(x) \leqslant \varepsilon^2 + \int\limits_{|x|>\varepsilon} x^2 dF_{nk}(x)$$

The last item agrees (6) at raither big n is evenly relative ε^2. Thus, for rather large n, evenly on relatively k and t in any final interval $|t| < T$.

$$|f_{nk}(t) - 1| \leqslant \varepsilon^2 T^2.$$

From here we conclude that is evenly relative k

$$\lim_{n\to\infty} f_{nk}(t) = 1. \tag{7}$$

From this is follows that for rather large n and t, belonging to a terminating interval $|t| < T$, inequality is carried out

$$|f_{nk}(t) - 1| < \frac{1}{2} \tag{8}$$

using decomposition of logarithm, we have

$$\ln \varphi_n(t) = \sum_{k=1}^{n} \ln f_{nk}(t) = \sum_{k=1}^{n} \ln[1 + (f_{nk}(t) - 1)] = \sum_{k=1}^{n} (f_{nk}(t) - 1) + R_n,$$

where

$$R_n = \sum_{k=1}^{n} \sum_{s=2}^{\infty} \frac{(-1)^s}{s} (f_{nk}(t) - 1)^s.$$

on the basis of (8)

$$|R_n| \leqslant \sum_{k=1}^{n} \sum_{s=2}^{\infty} \frac{1}{2} |f_{nk}(t) - 1|^s = \frac{1}{2} \sum_{k=1}^{n} \frac{|f_{nk}(t) - 1|^2}{1 - |f_{nk}(t) - 1|} \leqslant \sum_{k=1}^{n} |f_{nk}(t) - 1|^2.$$

As

$$\sum_{k=1}^{n}|f_{nk}(t)-1|=\sum_{k=1}^{n}\left|\int_{-\infty}^{\infty}(e^{itx}-1-itx)dF_{nk}(x)\right|\leqslant\frac{t^2}{2}\sum_{k=1}^{n}\int_{-\infty}^{\infty}x^2dF_{nk}(x)=\frac{t^2}{2},$$

that

$$|R_n|\leqslant\frac{t^2}{2}\max_{1\leqslant k\leqslant n}|f_{nk}(t)-1|.$$

From (7) follows

$$R_n\to 0.$$

Further,

$$\sum_{k=1}^{n}(f_{nk}(t)-1)=-\frac{t^2}{2}+\rho_n,$$

where

$$\rho_n=\frac{t^2}{2}+\sum_{k=1}^{n}\int_{-\infty}^{\infty}(e^{itx}-1-itx)dF_{nk}(x).$$

On the basis of (5)

$$\rho_n=\sum_{k=1}^{n}\int_{|x|<\varepsilon}\left(e^{itx}-1-itx+\frac{(tx)^2}{2}\right)dF_{nk}(x)+\sum_{k=1}^{n}\int_{|x|\geqslant\varepsilon}\left(\frac{t^2x^2}{2}+e^{itx}-1-itx\right)dF_{nk}(x).$$

Inequalities (1) and (2) allow to make estimates.

$$|\rho_n|\leqslant\frac{|t|^3}{6}\sum_{k=1}^{n}\int_{|x|\leqslant\varepsilon}|x|^3dF_{nk}(x)+t^2\sum_{k=1}^{n}\int_{|x|>\varepsilon}x^2dF_{nk}(x)\leqslant\frac{|t|^3}{6}\varepsilon\sum_{k=1}^{n}\int_{|x|\leqslant\varepsilon}x^2dF_{nk}(x)+$$

$$+t^2\sum_{k=1}^{n}\int_{|x|>\varepsilon}x^2dF_{nk}(x)=\frac{|t|^3}{6}+t^2\sum_{k=1}^{n}\int_{|x|>\varepsilon}x^2dF_{nk}(x).$$

On the basis of Lindeberg's condition, the right member of the last expression at $n\to\infty$ aspire to zero. Finely we have

$$\lim_{n\to\infty}\varphi_n(t)=e^{-\frac{t^2}{2}}.$$

Lectures 13–14

Statement of question.

The integral theorem of Moivre-Laplace was the ferst version of the central limit theorem. It is known that the integral theorem of Moivre-Laplace was a consequence of the local theorem, for the local theorem, for probabilities of a binomial distribution. Let's remind in brief these theorems. Let there is a sequence of the distributed random values indapendant equally $\xi_1,\dots,\xi_n$ with two outcomes

$$P(\xi_k=1)=p,\quad P(\xi_k=0)=q=1-p.$$

Let's consider the sum $S_n = \xi_+\xi_2 + \cdots + \xi_n$. Probability distribution S_n is defined by a binomial distribution,

$$P(S_n = m) = \frac{n!}{m!(n-m)!} p^m q^{n-m}.$$

The local theorem for a binomial distribution looks as follows, At $n \to \infty$ and $x = o(n^{1/6})$

$$P(S_n = m) = \frac{1}{\sqrt{2\pi npq}} e^{-\frac{x^2}{2}} \left\{ 1 + \frac{q-p}{6\sqrt{npq}}(x^3 - 3x) + O\left(\frac{1}{n}\right) \right\}.$$

Important point of this result is that the right member of the previous ratio is that the right member of the previous ratio is an item of the integral sum of Riemann, i.e.

$$\begin{aligned} \sum_{a<m<b} P(S_n = m) =& \frac{1}{\sqrt{2\pi}} \sum_{A<x<B} e^{-\frac{x^2}{2}} \Delta x \left\{ 1 + \frac{q-p}{6\sqrt{npq}}(x^3 - 3x) + \ldots \right\} \\ =& \frac{1}{\sqrt{2\pi}} \int_A^B e^{-\frac{x^2}{2}} dx + O\left(\frac{1}{\sqrt{n}}\right). \end{aligned} \tag{9}$$

where $\Delta x = \frac{1}{\sqrt{npq}}$, $\quad A = \frac{a-np}{\sqrt{npq}}$, $\quad B = \frac{b-np}{\sqrt{npq}}$.

Generalization of a binomial case.

Let $\xi_1, \ldots, \xi_k$ sequence of the distributed random variables independent equally with probability distribution

$$P(\xi_j = \nu) = p_\nu, \quad \sum_{\nu=1}^{k} p_\nu = 1$$

Let

$$S_n = \xi_1 + \cdots + \xi_n \in \{n, n+1, \ldots, kn\}$$

The task consists in finding a formula for probabilities $P(S_n = m)$.

For finding of the specified formula, we will use expression for characteristic function of size S_n

$$f_n(t) = \sum_{m=n}^{nk} p_m e^{itm}, \tag{10}$$

$$f_n(t) = \left(p_1 e^{it\cdot 1} + p_2 e^{it\cdot 2} + \cdots + p_k e^{it\cdot k}\right)^n = f^n(t). \tag{11}$$

Multiplying both parts of a ratio (10) on $\frac{1}{2\pi} e^{-itN}$ and integrating on an interval $(-\pi, \pi)$ we will recieve,

$$P(S_n = N) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_n(t) e^{-itN} dt.$$

On the basis of (11), after an involution, from the previous we will recieve

$$\begin{aligned} P(S_n = N) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f^n(t)e^{-itN}dt = \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{m_1+\dots+m_k=n} \frac{n!}{m_1!\dots m_k!} p_1^{m_1}\dots p_k^{m_k} e^{-it(N-m_1-2m_2-\dots-km_k)}dt. \end{aligned} \tag{12}$$

Relying on orthogonal property, the previous equally can be written down in a look

$$P(S_n = N) = \sum_{\substack{m_1+\dots+m_k=n \\ m_1+2m_2+\dots+km_k}} \frac{n!}{m_1!\dots m_k!}\, p_1^{m_1}\dots p_k^{m_k} \tag{13}$$

The task consists in studying of the asymptotic behavior of probabilities $P(S_n = N)$, at $n \to \infty$. The task is bound to the asymptotic behavior of probabilities of a polynomial distribution. On the basis of two representaitens of requirend probability formules (12) and (13), it is possible to consider two ways for a conclusion of local probability $P(S_n = m)$.

Polinomial distribution law. Let's consider serial independent tests, in each of which there is one of events $A_1, \dots, A_k$, with probabilities

$$P(A_i) = p_i, \quad p_1 + \dots + p_k = 1.$$

Will define probability of that in n tests the event A_1 will appear m_1 time, an event A_2 will appear m_2 time, ets. the event A_k will appear m_k time. Possible outcomes n test are various sets of events $A_{j_1}, A_{j_2}, \dots, A_{j_n}$ j_i, independently from each other can accept one of values $1, \dots, k$. Each such set represents the simple event, and their set makes space of the simple events. Let A_1 appears m_1 time, $\dots A_k$ appears m_k time. Probability of such event it will be equal

$$P(A_{j_1}, A_{j_2}, \dots, A_{j_k}) = p_1^{m_1}p_2^{m_2}\dots p_k^{m_k}$$

Number of all such probabilities to equally multinomial coefficient, therefore

$$P_n(m_1, \dots, m_k) = \frac{n!}{m_1!\dots m_k!}\, p_1^{m_1}\dots p_k^{m_k}. \tag{14}$$

Really, all shifts from n events equally $n!$. From them shifts formed by identical events equally $m_1!\dots m_k!$. Having reduced $n!$ by number of identical events we will recive the necessary resalt. In confirnmation, we will write down a know formula.

$$\sum_{m_1+\dots+m_k=n} \frac{n!}{m_1!\dots m_k!}\, p_1^{m_k}\dots p_k^{m_k} = (p_1 + \dots + p_k)^n = 1.$$

Models bringing to polynomial distribution One of them is selection with return of balls from a ballot box. Let in a ballot box are available a finite number of the numbered balls, numbers change from 1 to k. Thus the probability of emergence of a ball with number i is supposed equal p_i, $(p_1 + \dots + p_k = 1)$. Selection with volume return is made n. Then the probability of that balls with number 1 will appear m_1 time,..., with number k will appear m_k time, $(m_1 + \dots + m_k = n)$ are defined by a polenomeal distribution.

The following widespread model, distribution of particles on cells, is. Is available k sells, in each of which independently particulars from each other are in a random way distributed. The probability of hit of a particle in i a sell is identical to all particles and is equal p_i, $(p_1 + \dots + p_k = 1)$. Distribution n particles on sells is identical to all particles and is equal. In this case, the probability of that will get to the first cell m_1, ..., in k cell will get m_k also is defined by a polynomial distribution. The following model is bound to random values. Let $X_1, \dots, X_n$ sequence of the distributed random vectors independent equally. Let's consider unit vectors of Evklidov's space: $e_1 = (1, 0, \dots, 0), \dots, e_k = (0, \dots, 0, 1)$. In this case we will put that vectors X_i have the following probability distribution,

$$P(X_i = e_j) = p_j, \quad (i = 1, \dots, n, \quad p_1 + \dots + p_k = 1).$$

For descriptive reasons, we will give the following representation

$$X_i = \begin{cases} e_1 = (1, 0, 0, \dots, 0, 0), & \text{with probability } p_1, \\ e_2 = (0, 1, 0, \dots, 0, 0), & \text{with probability } p_2, \\ - - - - - - - - - & \\ e_k = (0, 0, 0, \dots, 0, 1), & \text{with probability } p_k. \end{cases} \tag{15}$$

If to consider the sum $X = X_1 + \dots + X_n$, in the assumption of that the first possible vector e_1 will appear m_1 time, the second it be shown m_2 time, etc., k-the possible vector will be shown m_k time, probability of a cooperative vector, $P\{X = (m_1, \dots, m_k)\}$, where $m_i \geq 0$ $m_1 + \dots + m_k = n$, whole will have a polynomial distribution.

Lectures 15–16

Properties of the probability distribution of a polinomial. We also give a formula for the probability distribution of the polinomial (14). For this we consider the transformed polinomial distribution probability formula.

In this terms, the following assertion holds:

$$n_i = m_i + \dots + m_k, \quad u_i = p_i + \dots + p_k, \quad q_i = \frac{p_i}{u_i}.$$

Lemma. The probabilities of the multinomial distribution can be represented as a product of conditional binomial probabilities

$$P_n(m_1, \dots, m_k) = \prod_{i=1}^{k-1} \frac{n_i!}{m_i!(n_i - m_i)!} q_i^{m_i} (1 - q_i)^{n_i - m_i}. \tag{16}$$

The validity of the previous formula, verifie the dusclosure of the right side and the reduction on the corresponding factors. Indeed, discributing the product and considering the factors individually obtain

$$\frac{n_1!}{m_1! n_2!} \frac{n_2!}{m_2! n_3!} \cdots \frac{n_{k-1}!}{m_{k-1}! n_k!} = \frac{n!}{m_1! \dots m_{k-1}! m_k!}$$

$$p_1^{m_1} \dots p_{k-1}^{m_{k-1}} p_k^{m_k} = p_1^{m_1} (1 - p_1)^{n - m_1} \left(\frac{p_2}{1 - p_1} \right)^{m_2} \left(1 - \frac{p_2}{1 - p_1} \right)^{n - m_1 - m_2} \cdots$$

$$\left(\frac{p_{k-1}}{1 - p_1 - \dots - p_{k-2}} \right)^{m_{k-1}} \left(1 - \frac{p_{k-1}}{1 - p_1 - \dots - p_{k-2}} \right)^{n - m_1 - \dots - m_{k-1}}$$

Size n_i has binomial distribution law, i.e.

$$P_n(n_i) = \frac{n!}{n_i!(n-n_i)!} u_i^{n_i}(1-u_i)^{n-n_i}, \quad n_i = \sum_{j=i}^{k} m_j, \quad u_i = \sum_{j=1}^{k} p_j. \tag{17}$$

Proof. Так как $n = m_1 + \cdots + m_k$, то при фиксированных $m_i, \ldots, m_k$ будем иметь,

$$P_n(n_i) = \sum_{\substack{m_1+\cdots+m_k=n \\ m_1+\cdots+m_{i-1}=n-n_i}} P_n(m_1, \ldots, m_k). \tag{18}$$

The following representation of a polynomial distribution is apparent

$$P_n(m_, \ldots, m_k) = \frac{n!}{m_i! \ldots m_k!(n-n_i)!}\, p_i^{m_i} \ldots p_k^{m_k}\, (1-u_i)^{n-n_i} \times$$
$$\frac{(n-n_i)!}{m_1! \ldots m_{i-1}!}\Big(\frac{p_1}{1-u_i}\Big)^{m_1} \cdots \Big(\frac{p_{i-1}}{1-u_i}\Big)^{m_k} \tag{19}$$

On the basis of the last expression, summing in (18) on $m_1, \ldots, m_{i-1}$, at the fixed $m_i, \ldots, m_k$ results, in the following equality,

$$P_n(n_i) = \sum_{m_i+\cdots+m_k=n_i} \frac{n!}{m_i! \ldots m_k!(n-m_i-\cdots-m_k)!} p_i^{m_i} \ldots p_k^{m_k}(1-u_i)^{n-m_i- \;\cdots - m_k}$$

Entering similarly (19) finally we will receive,

$$P_n(n_i) = \sum_{m_i+\cdots+m_k=n_i} \frac{n!}{n_i!(n-n_i)!} u_i^{n_i}(1-u_i)^{n-n_i} \times$$
$$\times \frac{n_i!}{m_i! \ldots m_k!}\Big(\frac{p_i}{u_i}\Big)^{m_i} \cdots \Big(\frac{p_k}{u_i}\Big)^{m_k} = \frac{n!}{n_i!(n-n_i)!} u_i^{n_i}(1-u_i)^{n-n_i}.$$

Moments of components of random vector in (15) equal,

$$Mm_i = np_i, \quad Mm_i^2 = n(n-1)p_i^2 + np_i, \quad Mm_i m_j = n(n-1)p_i p_j, \quad (i \neq j). \tag{20}$$

Proof.

$$Mm_1 = \sum_{m_1+\cdots+m_k=n} m_1 \frac{n!}{m_1! \ldots m_k!}\, p_1^{m_1} \ldots p_k^{m_k} =$$
$$= np_1 \sum_{(m_1-1)+m_2+\cdots+m_k=n-1} \frac{(n-1)!}{(m_1-1)!\, m_2! \ldots m_k!}\, p_1^{m_1-1} \ldots p_k^{m_k} = np_1$$

It is apparent that the previous ratio is fair and for the arbitraries m_i. futher,

$$Mm_1(m_1-1) = \sum_{m_1+\cdots+m_k=n} m_1(m_1-1) \frac{n!}{m_1! \ldots m_k!}\, p_1^{m_1} \ldots p_k^{m_k} =$$
$$= n(n-1)p_1^2 \sum_{(m_1-2)+m_2+\cdots+m_k=n-2} \frac{(n-2)!}{(m_1-2)!\, m_2! \ldots m_k!}\, p_1^{m_1-1} \ldots p_k^{m_k} = n(n-1)p_1^2$$

$$Mm_1m_2 = \sum_{m_1+\dots+m_k=n} m_1m_2\frac{n!}{m_1!\dots m_k!}\,p_1^{m_1}\dots p_k^{m_k} =$$

$$= n(n-1)p_1^2 \sum_{(m_1-1)+(m_2-1)+\dots+m_k=n-2} \frac{(n-2)!}{(m_1-1)!\,(m_2-1)!\,m_3!\dots m_k!}\,p_1^{m_1-1}p_2^{m_2-1}\dots p_k^{m_k} =$$

$$= n(n-1)p_1p_2$$

Thereby the lemma is proved.

Lectures 17 – 18

Pearson's local theorem. t $n \to \infty$, $x_i = o(n^{1/6})$ for probabilities of a polinomial distribution fairly following asimptotic decomposition

$$P_n(m_1,\dots,m_k) = \frac{1}{(\sqrt{2\pi n})^{k-1}\sqrt{p_1\dots p_k}}e^{-\frac{1}{2}\sum\limits_{i=1}^{k}x_i^2}\left\{1+\sum_{i=1}^{k}\frac{x_i^3-3x_i}{6\sqrt{np_i}}+O\left(\frac{1}{n}\right)\right\} \qquad (21)$$

,

where $x_i = \frac{m_i-np_i}{\sqrt{np_i}}$.

Proof. Let's use a Stirling formula

$$n! = \sqrt{2\pi}\;n^{n+1/2}e^{-n} + \frac{\theta}{12n} \quad (0<\theta<1)$$

All factorials in (22) it is replaceble on a Stirling formula. As a result we will receive,

$$P_n(m_1,\dots,m_k) = \frac{n^{n+\frac{1}{2}}e^{-n}\;p_1^{m_1}\dots p_k^{m_k}}{\left(\sqrt{2\pi n}\right)^{k-1}\;m_1^{m_1+\frac{1}{2}}e^{-m_1}\dots{}_k^{m_k+\frac{1}{2}}\;e^{-m_k}}\exp\left\{\frac{\theta}{12n}-\frac{\theta_1}{12m_1}-\dots-\frac{\theta_k}{12m_k}\right\}$$

The received expression can be written down in the following

$$P_n(m_1,\dots,m_k) = \frac{1}{\left(\sqrt{2\pi n}\right)^{k-1}\;\sqrt{p_1\dots p_k}}\left(\frac{m_1}{np_1}\right)^{-(m_1+\frac{1}{2})}\dots\left(\frac{m_k}{np_k}\right)^{-(m_k+\frac{1}{2})}$$
$$\exp\left\{\frac{\theta}{12n}-\frac{\theta_1}{12m_1}-\dots-\frac{\theta_k}{12m_k}\right\} = H_1H_2H_3.$$

where

$$H_1 = \frac{1}{\left(\sqrt{2\pi n}\right)^{k-1}\;\sqrt{p_1\dots p_k}}, \quad H_2 = \left(\frac{m_1}{np_1}\right)^{-(m_1+\frac{1}{2})}\dots\left(\frac{m_k}{np_k}\right)^{-(m_k+\frac{1}{2})}$$

$$H_3 = \exp\left\{\frac{\theta}{12n}-\frac{\theta_1}{12m_1}-\dots-\frac{\theta_k}{12m_k}\right\}$$

Substituting $m_i = np_i + x_i\sqrt{np_i}$ and substituting its expression for $\ln H_2$ we will receive,

$$\ln H_2 = -(np_1+x_1\sqrt{np_1}+1/2)\ln\left(1+\frac{x_1}{\sqrt{np_1}}\right)-\dots-(np_k+x_k\sqrt{np_k}+1/2)\ln\left(1+\frac{x_k}{\sqrt{np_k}}\right)$$

Decompositing logarithms in a row, removing the brackets and giving the corresponding expressions at identical degrees $(np_i)^{-j}$ we will receive

$$\ln H_2 = -\frac{1}{2}\sum_{i=1}^{k} x_i^2 + R(x_1, \ldots, x_k).$$

where

$$R(x_1, \ldots, x_k) = \sum_{i=1}^{k}\left\{\sum_{j=1}^{\infty}(-1)^{j+1}\frac{x_i^{j+2}}{(j+1)(j+2)(np_i)^{j/2}} + \frac{1}{2}\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\left(\frac{x_i}{\sqrt{np_i}}\right)^j\right\}$$

For $\ln H_2$ we will be limitied to two terms, of decomposition, i.e.

$$R(x_1, \ldots, x_k) = -\frac{1}{2}\sum_{i=1}^{k} x_i^2 + \sum_{i=1}^{k}\frac{x_i^3 - 3x_i}{6\sqrt{np_i}} + O\left(\frac{1}{n}\right).$$

Using the received representation, in the form of the Gram-Charlier expansion in a series, we will receive (22).

Remarks.

1) From Pirson's theorem at $k = 2$ the theorem of Moivre does not follows. This results from the fact that normalization in the corresponding theorems a various. Really, in the theorem Moivre the normalization has an appearence

$$x = \frac{m - np}{\sqrt{npq}}$$

In Pearson's theorem

$$x_i = \frac{m_i - np_i}{\sqrt{np_i}}.$$

2) In this regard, from Pearson's decomposition, it is impossible immediatly, to pass to the integral theorem. The matter is that expression

$$\chi^2 = \sum_{i=1}^{k}\frac{(m_i - np_i)^2}{np_i} = \sum_{i=1}^{k} x_i^2$$

represents the positive definet quadratic form.It follows from a condition,

$$\sum_{i=1}^{k} x_i\sqrt{p_i} = \sum_{i=1}^{k}\frac{m_i - np_i}{\sqrt{n}} = 0.$$

Thus, we are faced by task — to consider Pearson's local theorem in the transformed form. It can be reached by means of reduction of the quadratic form to a canonical form. Such transformation is Helmert's generalized transformation.

Lectures 19—20.

Helmert's generalized transformation.

Let there is a quadratic form χ^2, presented in a look

$$\chi^2 = x_1^2 + x_2^2 + \cdots + x_k^2; \quad c_1x_1 + c_2x_2 + \cdots + c_kx_k = \lambda, \tag{22}$$

where $\lambda, c_i, \quad i = 1, 2, \ldots, k$ real numbers.

Let's enter designations: $\vec{X} = (x_1, \ldots, x_k)^T, \quad \vec{Y} = (\lambda/\omega_1, y_1, \ldots, y_{k-1})^T$, $\omega_i^2 = c_i^2 + c_{i+1}^2 + \cdots + c_k^2, \quad \vec{c} = (c_1/\omega_1, c_2/\omega_1, \ldots, c_k/\omega_1)^T$

Theorema.

The quadratic form (1), by means of orthogonal transformation $\vec{X} = C_1\vec{Y}$, где

$$C_1 = \begin{pmatrix}
\frac{c_1}{\omega_1} & \pm\frac{\omega_2}{\omega_1} & 0 & 0 & \cdots & 0 & 0 \\
\frac{c_2}{\omega_1} & \mp\frac{c_1c_2}{\omega_1\omega_2} & \pm\frac{\omega_3}{\omega_2} & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{c_{k-2}}{\omega_1} & \mp\frac{c_1c_{k-2}}{\omega_1\omega_2} & \mp\frac{c_2c_3}{\omega_2\omega_3} & \mp\frac{c_3c_4}{\omega_3\omega_4} & \cdots & \pm\frac{\omega_{k-1}}{\omega_{k-2}} & 0 \\
\frac{c_{k-1}}{\omega_1} & \mp\frac{c_1c_{k-1}}{\omega_1\omega_2} & \mp\frac{c_2c_{k-1}}{\omega_2\omega_3} & \mp\frac{c_3c_{k-1}}{\omega_3\omega_4} & \cdots & \mp\frac{c_{k-2}c_{k-1}}{\omega_{k-2}\omega_{k-1}} & \pm\frac{\omega_k}{\omega_{k-1}} \\
\frac{c_k}{\omega_1} & \mp\frac{c_1c_k}{\omega_1\omega_2} & \mp\frac{c_2c_k}{\omega_2\omega_3} & \mp\frac{c_3c_k}{\omega_3\omega_4} & \cdots \mp\frac{c_{k-2}c_k}{\omega_{k-2}\omega_{k-1}} & \mp\frac{c_{k-1}c_k}{\omega_{k-1}\omega_k} &
\end{pmatrix}$$

is provided to a look $\chi^2 = \lambda^2/\omega_1^2 + y_1^2 + \cdots + y_{k-1}^2$.

The generalized transformation is necessary for a special case which meets in the multidimensional local theorem of Moivre-Laplace

$$\chi_1^2 = x_1^2 + x_2^2 + \cdots + x_k^2; \quad \sqrt{p_1}x_1 + \sqrt{p_2}x_2 + \cdots + \sqrt{p_k}x_k = 0, \tag{23}$$

where $p_i > 0, p_1 + p_2 + \cdots + p_k = 1$. That transformation of the quadratic form (2) has a degenerate matrix is remarkable. Nevertheless, this transformation leads χ_1^2 to a look: $\chi_1^2 = y_1^2 + y_2^2 + \cdots + y_{k-1}^2$.

Lectures 21 — 22.

Multidimensional of Moivre – Laplace theorem.

Let's consider expression of asymptotic decomposition for probabilities of a polynomialdistribution

$$P_n(m_1, \ldots, m_k) = \frac{1}{(\sqrt{2\pi n})^{k-1}\sqrt{p_1 \ldots p_k}} e^{-\frac{1}{2}\sum\limits_{i=1}^{k} x_i^2} \left\{1 + \sum_{i=1}^{k} \frac{x_i^3 - 3x_i}{6\sqrt{np_i}} + O\left(\frac{1}{n}\right)\right\} \tag{24}$$

,

where $x_i = \frac{m_i - np_i}{\sqrt{np_i}}$.

We will write out expression of the quadratic form in the following form:

$$\chi^2 = x_1^2 + x_2^2 + \cdots + x_k^2; \quad \sqrt{p_1}x_1 + \sqrt{p_2}x_2 + \cdots + \sqrt{p_k}x_k = 0, \tag{25}$$

On the basis a lemma 1 we will construct transformation

$$
\begin{array}{lll}
x_1 & = & \sqrt{\frac{u_2}{u_1}} y_1 \\
x_2 & = -\sqrt{\frac{p_1 p_2}{u_1 u_2}} y_1 & + \sqrt{\frac{u_3}{u_2}} y_2 \\
x_3 & = -\sqrt{\frac{p_1 p_3}{u_1 u_2}} y_1 & - \sqrt{\frac{p_2 p_3}{u_2 u_3}} y_2 \quad + \quad \sqrt{\frac{u_4}{u_3}} \\
- \\
x_{k-1} & = -\sqrt{\frac{p_1 p_{k-1}}{u_1 u_2}} y_1 - \sqrt{\frac{p_2 p_{k-1}}{u_2 u_3}} y_2 - \cdots - \sqrt{\frac{p_{k-2} p_{k-1}}{u_{k-2} u_{k-1}}} y_{k-2} + \sqrt{\frac{u_k}{u_{k-1}}} y_{k-1} \\
x_k & = -\sqrt{\frac{p_1 p_k}{u_1 u_2}} y_1 & - \sqrt{\frac{p_2 p_k}{u_2 u_3}} y_2 \quad - \cdots - \sqrt{\frac{p_{k-2} p_k}{u_{k-2} u_{k-1}}} y_{k-2} - \sqrt{\frac{p_{k-1} p_k}{u_{k-1} u_k}} y_{k-1}
\end{array}
\tag{26}
$$

where

$$u_i = p_i + \cdots + p_k, \quad u_1 = 1, \quad u_k = p_k.$$

The matrix of transformation (27) is degenerate is remarkable. Nevertheless we can define an inverse transformation. The new variable will have an appearence

$$
\begin{aligned}
& y_i = x_i \sqrt{\frac{u_{i+1}}{u_i}} - \sqrt{\frac{p_i}{u_i u_{i+1}}} \Big(x_{i+1} \sqrt{p_{i+1}} + \cdots + x_k \sqrt{p_k} \Big) = \\
& \frac{m_i - np_i}{np_i} \sqrt{\frac{u_{i+1}}{u_i}} - \sqrt{\frac{p_i}{u_i u_{i+1}}} \; \frac{m_{i+1} + \cdots + m_k - n(p_{i+1} + \cdots + p_k)}{n} = \\
& \frac{1}{\sqrt{np_i u_i u_{i+1}}} \Big[(m_i - np_i) u_{i+1} - p_i (m_{i+1} + \cdots + m_k - n u_{i+1}) \Big] = \\
& \frac{1}{\sqrt{np_i u_i u_{i+1}}} \Big[(m_i - np_i) u_{i+1} + m_i p_i - p_i (m_i + m_{i+1} + \cdots + m_k - n u_{i+1}) \Big].
\end{aligned}
$$

Thus,

$$y_i = \frac{m_i u_i - (n - m_1 - \cdots - m_{i-1}) p_i}{\sqrt{np_i u_i u_{i+1}}}. \quad (i = 1, 2, \ldots, k-1). \tag{27}$$

On the basis of transformation (27) expression (25) will assume an air:

$$P_n(m_1, \ldots, m_k) = \frac{1}{(\sqrt{2\pi n})^{k-1} \sqrt{p_1 \ldots p_k}} e^{-\frac{1}{2} \sum\limits_{i=1}^{k-1} y_i^2} \left\{ 1 + \sum_{i=1}^{k} \frac{x_i^3 - 3x_i}{6\sqrt{np_i}} + O\left(\frac{1}{n}\right) \right\} \tag{28}$$

, Let's consider an increase y_i, at the fixed values $m_1, \ldots, m_{i-1}$. As a result, we will receve

$$\Delta y_i = y_i(m_i + 1) - y_i(m_i) = \sqrt{\frac{u_i}{np_i u_{i+1}}}.$$

Therefore,

$$\Delta y_1 \ldots \Delta y_{k-1} = \frac{1}{(\sqrt{n})^{k-1} \sqrt{p_1 \ldots p_k}}.$$

By means of transformation (27) we will receive,

$$\sum_{i=1}^{k}\frac{x_i^3-3x_i}{6\sqrt{np_i}}=\sum_{i=1}^{k-1}\frac{1}{6\sqrt{np_iu_iu_{i+1}}}\left\{(u_{i+1}-p_i)y_i^3-3y_i\left[u_{i+1}-(k-i)p_i+p_i\sum_{j=i+1}^{k-1}y_i^2\right]\right\}. \quad (29)$$

where in a right member of the last expression, $i=1,\dots,k-1$, $\sum\limits_{2}^{1}\equiv 0$. Let's notice that at $k=2$, the ratio (30) will assume an air,

$$\sum_{i=1}^{2}\frac{x_i^3-3x_i}{6\sqrt{np_i}}=\frac{q-p}{6\sqrt{npq}}(x^3-3x).$$

So, decomposition (29) sign in forme

$$\begin{aligned}P_n(m_1,\dots,m_k)&=\frac{1}{(\sqrt{2\pi})^{k-1}}e^{\frac{1}{2}\sum\limits_{i=1}^{k-1}y_i^2}\Delta y_1\dots\Delta y_{k-1}\times\\ &\times\left\{1+\sum_{i=1}^{k-1}\frac{1}{6\sqrt{np_iu_iu_{i+1}}}\left\{(u_{i+1}-p_i)y_i^3-3y_i\left[u_{i+1}-(k-i)p_i+p_i\sum_{j=i+1}^{k-1}y_i^2\right]\right\}\right\}.\end{aligned} \quad (30)$$

Let's consider a body (31) in a look,

$$\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(m_i-(n-m_1-\dots-m_{i-1})p_i)^2}{2np_i\frac{u_{i+1}}{u_i}}\right\}\frac{1}{\sqrt{np_i\frac{u_{i+1}}{u_i}}}.$$

Thus size m_i at fixed $m_1,\dots m_{i-1}$ has the normal distribution law with parameters $a=(n-m_1-\dots-m_{i-1})p_i$, $\sigma^2=np_i\frac{u_{i+1}}{u_i}$.

Lecture 23—24.

Culkulater of moments y_i.

For finding of unknown quantaties of the polinomial distribution law.

$$Mm_i=np_i,\quad Mm_i^2=n(n-1)p_i^2+np_i,\quad Mm_im_j=n(n-1)p_ip_j\ \ (i\neq j).$$

Let's enter designation $n_i=m_i+\dots+m_k$.

$$Mn_i=nu_i,\quad Mn_i^2=n(n-1)u_i^2+nu_i.$$

Let's consider also at, $i<j$.

$$Mn_in_j=M(m_i+\dots+m_{j-1}+n_j)n_j=n(n-1)(p_i+\dots+p_{j-1})u_j+n(n-1)u_j^2+nu_j=$$

$$=n(n-1)(p_i+\dots+p_{j-1}+u_j)u_j+nu_j=n(n-1)u_iu_j+nu_j.$$

Let's y_i consider representation

$$y_i=\frac{m_iu_{i+1}-(n-m_1-\dots-m_i)p_i}{\sqrt{np_iu_iu_{i+1}}}=\frac{m_iu_{i+1}-n_{i+1}p_i}{\sqrt{np_iu_iu_{i+1}}}$$

Follows,

$$My_i = \frac{np_iu_{i+1} - np_iu_{i+1}}{\sqrt{np_iu_iu_{i+1}}} = 0, \qquad Dy_i = My_i^2 = \frac{m_i^2u_{i+1}^2 - 2m_in_{i+1}u_{i+1}p_i + n_{i+1}^2p_i^2}{np_iu_iu_{i+1}} =$$

$$= \frac{1}{np_iu_iu_{i+1}}\left[n(n-1)p_i^2u_{i+1}^2 + np_iu_{i+1}^2 - 2n(n-1)u_{i+1}^2p_i^2 + n(n-1)u_{i+1}^2p_i^2 + nu_{i+i}p_i^2\right] =$$

$$= \frac{nu_{i+1}p_i(p_i+u_{i+1})}{np_iu_iu_{i+1}} = \frac{np_iu_iu_{i+1}}{np_iu_iu_{i+1}} = 1$$

Beliving $i < j$,

$$Cov(y_i, y_j) = My_iy_j = M\left(\frac{m_iu_{i+1} - n_{i+1}p_i}{\sqrt{np_iu_iu_{i+1}}}\ \frac{m_ju_{j+1} - n_{j+1}p_j}{\sqrt{np_ju_ju_{j+1}}}\right)$$

Let's consider expectation

$$M(m_iu_{i+1} - n_{i+1}p_i)(m_ju_{j+1} - n_{j+1}p_j) = M(m_im_ju_{i+1}u_{j+1} - m_in_{j+1}u_{i+1}p_j -$$

$$-m_jn_{i+1}p_iu_{j+1} + n_{i+1}n_{j+1}p_ip_j) = n(n-1)\{p_ip_ju_{i+1}u_{j+1} - p_ip_ju_{i+1}u_{j+1} -$$

$$-p_ip_ju_{i+1}u_{j+1}\} - np_ip_ju_{j+1} + np_ip_ju_{j+1} = 0$$

Thus $y_i \quad (i = 1, \dots, k-1)$ is independent.

Integral theorems.

Let there is some squarable are G. It is reqred to find probability

$$P(\vec{m} \in G) = \sum_{\vec{m}\in G} \frac{n!}{m_1! \dots m_k!} p_i^{m_1} \dots p_k^{m_k}.$$

Let's apply the many-dimensional local theorem we have

$$P(\vec{m} \in G) = \frac{1}{(\sqrt{2\pi})^{k-1}} \sum_{\vec{m}\in G} e^{\frac{1}{2}\sum\limits_{i=1}^{k-1} y_i^2} \Delta y_1 \dots \Delta y_{k-1} \times$$

$$\times \left\{1 + \sum_{i=1}^{k-1} \frac{1}{6\sqrt{np_iu_iu_{i+1}}}\left\{(u_{i+1} - p_i)y_i^3 - 3y_i\left[u_{i+1} - (k-i)p_i + p_i\sum_{j=i+1}^{k-1} y_i^2\right]\right\}\right\}.$$

Apply a toting formula

$$P(\vec{m} \in G) = \frac{1}{(\sqrt{2\pi})^{k-1}} \int \dots \int_{\vec{y}\in G} e^{\frac{1}{2}\sum\limits_{i=1}^{k-1} y_i^2}\, dy_1 \dots dy_{k-1} + O\left(\frac{1}{\sqrt{n}}\right). \qquad (31)$$

Private cases: 1)Let

$$G = \left\{(y_1, \dots y_{k-1}) : \chi^2 = \sum_{i=1}^{k-1} y_i^2 \leqslant x\right\}.$$

In this case, we came to a chi-square of distribution,

$$P(\chi^2 \leqslant x) = \frac{1}{2^{\frac{k-1}{2}}\Gamma\left(\frac{k-1}{2}\right)} \int\limits_0^x z^{\frac{k-1}{2}-1} e^{-\frac{z}{2}} dz + \varepsilon_n,$$

$\varepsilon_n = O\left(\frac{1}{n}\right)$. 2)Let

$$G = \{a_1 < y_1 \leqslant b_1, \dots, a_{k-1} < y \leqslant b_{k-1}\}.$$

$$P(\vec{y} \in G) = \prod_{i=1}^{k-1} \left(\frac{1}{\sqrt{2\pi}} \int\limits_{a_i}^{b_i} e^{-\frac{y^2}{2}} dy_i \right) + O\left(\frac{1}{\sqrt{n}}\right).$$

3)Let

$$G = \left\{ (y_1, \dots y_{k-1}) : \max_{i \leqslant i \leqslant k-1} |y_i| \leqslant x \right\}.$$

$$P(\max_{i \leqslant i \leqslant k-1} |y_i| \leqslant x) = \left(\frac{1}{\sqrt{2\pi}} \int\limits_{a_i}^{b_i} e^{-\frac{y^2}{2}} dy_i \right)^{k-1} + O\left(\frac{1}{n}\right).$$

The last ratio is received by means a formula of toting of Eyler-Makloren.

Lecture 25-26.

Direct method of summation.

Let $\xi_1, \dots, \xi_k$ sequence of the distributed sizes independent equally with a probability distribution

$$P(\xi_j = \nu) = p_\nu, \quad \sum_{\nu=1}^{k} p_\nu = 1$$

Let

$$S_n = \xi_1 + \dots + \xi_n \in \{n, n+1, \dots, kn\}$$

We need to find a formula for probabilities $P(S_n = m)$. Let's use the following expression for characteristic function of S_n

$$f_n(t) = \sum_{m=n}^{nk} p_m e^{itm}, \tag{32}$$

$$f_n(t) = \left(p_1 e^{it\cdot 1} + p_2 e^{it\cdot 2} + \dots + p_k e^{it\cdot k}\right)^n = f^n(t). \tag{33}$$

Multiplying both parts of a ratio (10) non $\frac{1}{2\pi}e^{-itN}$ and integrating on an interval, $(-\pi, \pi)$,we will have

$$P(S_n = N) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f_n(t) e^{-itN} dt.$$

On the (11) we get

$$P(S_n = N) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^n(t)e^{-itN}dt = \\ = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{m_1+\dots+m_k=n} \frac{n!}{m_1!\dots m_k!} p_1^{m_1}\dots p_k^{m_k} e^{-it(N-m_1-2m_2-\dots-km_k)}dt. \tag{34}$$

Or

$$P(S_n = N) = \sum_{\substack{m_1+\dots+m_k=n \\ m_1+2m_2+\dots+km_k}} \frac{n!}{m_1!\dots m_k!}\, p_1^{m_1}\dots p_k^{m_k} \tag{35}$$

Using formulas (35) and (31) we get

$$P(S_n = N) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{m_1+\dots+m_k=n} \frac{e^{-\frac{1}{2}\sum\limits_{j=1}^{k} x_j^2 - it(N-m_1-2m_2-\dots-km_k)}}{(\sqrt{2\pi n})^{k-1}\sqrt{p_1\dots p_k}}dt \tag{36}$$

Using $m_j = np_j + x_j\sqrt{np_j}$, we get

$$-2it(m_1+2m_2+\dots+km_k-N) = 2it\left[N - na - \sqrt{n}\Big(x_1\sqrt{p_1} + 2x_2\sqrt{p_2} + \dots + kx_k\sqrt{p_k}\Big)\right].$$

Using of transformation,

$$\begin{aligned}
x_1 &= \sqrt{\frac{u_2}{u_1}}y_1 \\
x_2 &= -\sqrt{\frac{p_1p_2}{u_1u_2}}y_1 + \sqrt{\frac{u_3}{u_2}}y_2 \\
x_3 &= -\sqrt{\frac{p_1p_3}{u_1u_2}}y_1 - \sqrt{\frac{p_2p_3}{u_2u_3}}y_2 + \sqrt{\frac{u_4}{u_3}} \\
&- \\
x_{k-1} &= -\sqrt{\frac{p_1p_{k-1}}{u_1u_2}}y_1 - \sqrt{\frac{p_2p_{k-1}}{u_2u_3}}y_2 - \dots - \sqrt{\frac{p_{k-2}p_{k-1}}{u_{k-2}u_{k-1}}}y_{k-2} + \sqrt{\frac{u_k}{u_{k-1}}}y_{k-1} \\
x_k &= -\sqrt{\frac{p_1p_k}{u_1u_2}}y_1 - \sqrt{\frac{p_2p_k}{u_2u_3}}y_2 - \dots - \sqrt{\frac{p_{k-2}p_k}{u_{k-2}u_{k-1}}}y_{k-2} - \sqrt{\frac{p_{k-1}p_k}{u_{k-1}u_k}}y_{k-1}
\end{aligned}$$

get

$$x_1\sqrt{p_1} + 2x_2\sqrt{p_2} + \dots + kx_k\sqrt{p_k} = \sqrt{\frac{p_1}{u_1u_2}}(u_2 - 2p_2 - \dots - kp_k)y_1 + \sqrt{\frac{p_2}{u_2u_3}}(2u_3 - 3p_3 - \\ - \dots - kp_k)y_2 + \dots + \sqrt{\frac{p_{k-1}}{u_{k-1}u_k}}(u_k(k-1) - kp_k)y_{k-1} = -\sqrt{\frac{p_1}{u_1u_2}}a_1y_1 - \\ - \sqrt{\frac{p_2}{u_2u_3}}a_2y_2 - \dots - \sqrt{\frac{p_{k-1}}{u_{k-1}u_k}}a_{k-1}y_{k-1}.$$

where $a_j = p_{j+1} + 2p_{j+2} + \cdots + (k-j)p_k$. Consiquently,

$$x_1^2 + \cdots + x_k^2 - 2it(x_1\sqrt{p_1} + 2x_2\sqrt{p_2} + \cdots + kx_k\sqrt{p_k} = y_1^2 + \cdots + y_{k-1}^2 +$$

$$2it\left(\sqrt{\frac{p_1}{u_1u_2}}a_1y_1 + \sqrt{\frac{p_2}{u_2u_3}}a_2y_2 + \cdots + \sqrt{\frac{p_{k-1}}{u_{k-1}u_k}}a_{k-1}y_{k-1}\right) =$$

$$= \left(y_1 + it\sqrt{\frac{p_1}{u_1u_2}}a_1\right)^2 + \cdots + \left(y_{k-1} + it\sqrt{\frac{p_{k-1}}{u_{k-1}u_k}}a_{k-1}\right)^2 + t^2\sigma^2.$$

where $\sigma^2 = p_1 + 2^2p_2 + \cdots + k^2p_k - a^2 = M\xi^2 - (M\xi)^2 = D\xi$. On the basis

$$P(S_n = N) = \frac{1}{2\pi\sqrt{n}} \int\limits_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \sum_{\chi^2<r^2} \frac{\exp\left\{-\frac{1}{2}\sum\limits_{j=1}^{k-1}\left(y_j + it\sqrt{\frac{p_j}{u_ju_{j+1}}}a_j\right)^2 - \frac{t^2\sigma^2}{2} - it\xi\right\}}{(\sqrt{2\pi n})^{k-1}\sqrt{p_1\ldots p_k}} dt + O\left(\frac{1}{\sqrt{n}}\right)$$

where $\xi = \frac{N-na}{\sqrt{n}}$.

Lectures 27 — 28

Direct method (continuation).

Way 1.

Let's write down required probability, in a look

$$P(S_n = N) = \frac{1}{2\pi\sqrt{n}} \int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}(t^2\sigma^2 - 2it\xi)}\Lambda(y_1, \ldots, y_{k-1})dt + O\left(\frac{1}{\sqrt{n}}\right). \tag{37}$$

where

$$\Lambda(y_1, \ldots, y_{k-1}) = \frac{1}{(\sqrt{2\pi})^{k-1}} \sum_{\chi^2<r^2} \exp\left\{-\frac{1}{2}\sum_{j=1}^{k-1}\left(y_j + ita_j\sqrt{\frac{p_j}{u_ju_{j+1}}}\right)^2\right\}\Delta y_1 \ldots \Delta y_{k-1}.$$

As a result of summing the previous expression will assume an air,

$$\Lambda = \frac{1}{(\sqrt{2\pi})^{k-1}} \int\limits_{-\infty}^{\infty}\cdots\int \exp\left\{-\frac{1}{2}\sum_{j=1}^{k-1}\left(y_j + ita_j\sqrt{\frac{p_j}{u_ju_{j+1}}}\right)^2\right\} dy_1 \ldots dy_{k-1} + O\left(\frac{1}{\sqrt{n}}\right). \tag{38}$$

Let's make variable replacement

$$z_j = y_j + ita_j\sqrt{\frac{p_j}{u_ju_{j+1}}},$$

and having substituted integral in (39), we will receive

$$\Lambda = \frac{1}{(\sqrt{2\pi})^{k-1}} \int\limits_{-\infty}^{\infty}\cdots\int e^{-\frac{1}{2}\sum\limits_{j=1}^{k-1} z_j^2} dz_1 \ldots dz_{k-1} + O\left(\frac{1}{\sqrt{n}}\right) = 1 + O\left(\frac{1}{\sqrt{n}}\right)$$

Substituting the received result for Λ in (38), we will have

$$P(S_n = N) = \frac{1}{2\pi\sqrt{n}} \int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}(t^2\sigma^2 - 2it\xi)} dt + O\left(\frac{1}{\sqrt{n}}\right) =$$

$$= \frac{1}{2\pi\sqrt{n}} \int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}\left(t\sigma - \frac{i\xi}{\sigma}\right) - \frac{\xi^2}{2\sigma^2}} dt + O\left(\frac{1}{\sqrt{n}}\right).$$

After variable replacement

$$z = t\sigma - \frac{i\xi}{\sigma}, \quad dt = \frac{1}{\sigma} dz,$$

we get,

$$P(S_n = N) = \frac{1}{\sqrt{2\pi n}\sigma} e^{-\frac{(n-na)^2}{2n\sigma^2}} + O\left(\frac{1}{\sqrt{n}}\right).$$

Thus, we received the local theorem

Way 2.

Let's proceed from a formula (36)

$$P(S_n = N) = \sum_{\substack{m_1+\cdots+m_k=n \\ m_1+2m_2+\cdots+km_k}} \frac{n!}{m_1!\ldots m_k!}\, p_1^{m_1}\ldots p_k^{m_k}$$

We will need the following

Lemma

$$\begin{aligned} &\chi^2 = x_1^2 + x_2^2 + \cdots + x_k^2, \\ &x_1\sqrt{p_1} + x_2\sqrt{p_2} + \cdots + x_k\sqrt{p_k} = t \\ &x_1\sqrt{p_1} + 2x_2\sqrt{p_2} + \cdots + kx_k\sqrt{p_k} = \tau \end{aligned} \tag{39}$$

by means of orthogonal transformation (27) it is led to a look

$$\chi^2 = t^2 + \frac{(\tau - at)^2}{\sigma^2} + \sum_{\nu=1}^{k-2} z_\nu^2$$

where $a = p_1 + 2p_2 + \cdots + kp_k, \quad \sigma^2 = \sum\limits_{\nu=1}^{k} (\nu - a)^2 p_\nu$

Let's note the main stages of the proof. At the first stage we consider

$$\chi^2 = x_1^2 + x_2^2 + \cdots + x_k^2, \quad x_1\sqrt{p_1} + x_2\sqrt{p_2} + \cdots + x_k\sqrt{p_k} = t.$$

On the basis of (27), we built the following orthogonal transformation

$$
\begin{aligned}
x_1 &= \sqrt{\frac{u_2}{u_1}}y_1 + t\sqrt{p_1} \\
x_2 &= -\sqrt{\frac{p_1p_2}{u_1u_2}}y_1 + \sqrt{\frac{u_3}{u_2}}y_2 + t\sqrt{p_2} \\
&------------------------------ \\
x_{k-1} &= -\sqrt{\frac{p_1p_{k-1}}{u_1u_2}}y_1 - \cdots - \sqrt{\frac{p_{k-2}p_{k-1}}{u_{k-2}u_{k-1}}}y_{k-2} + \sqrt{\frac{u_k}{u_{k-1}}}y_{k-1} + t\sqrt{p_{k-1}} \\
x_k &= -\sqrt{\frac{p_1p_k}{u_1u_2}}y_1 - \cdots - \sqrt{\frac{p_{k-2}p_k}{u_{k-2}u_{k-1}}}y_{k-2} - \sqrt{\frac{p_{k-1}p_k}{u_{k-1}u_k}}y_{k-1} + t\sqrt{p_k}
\end{aligned}
\tag{40}
$$

Lectures 29 — 30
Continuation.

Using (40), we will find expression of the second linear condition from (39): in terms

$$\sum_{\nu=1}^{k} \nu x_\nu \sqrt{p_\nu} = \sum_{\nu=1}^{k-1} c_\nu y_\nu + at = \tau,$$

where

$$
\begin{aligned}
c_\nu &= \sqrt{\frac{u_{\nu+1}p_\nu}{u_\nu}}\nu - \sqrt{\frac{p_\nu}{u_\nu u_{\nu+1}}}\Big((\nu+1)p_{\nu+1} + (\nu+2)p_{\nu+2} + \cdots + kp_k\Big) = \\
&= -\sqrt{\frac{p_{nu}}{u_\nu u_{\nu+1}}}\Big(p_{\nu+1} + 2p_{\nu+2} + \cdots + (k-\nu)p_\nu\Big) = -\sqrt{\frac{p_\nu}{u_\nu u_{\nu+1}}}\ a_{\nu+1} \\
a_\nu &= p_\nu + 2p_{\nu+1} + \cdots + (k-\nu+1)p_k.
\end{aligned}
\tag{41}
$$

Noticing, that $a_{\nu+1} - a_{\nu+2} = u_{\nu+1}$ we will have

$$
\begin{aligned}
\sigma_\nu^2 &= c_\nu^2 + \cdots + c_k^2 = \frac{p_\nu}{u_\nu u_{\nu+1}}a_{\nu+1}^2 + \frac{p_{\nu+1}}{u_{\nu+1}u_{\nu+2}}a_{\nu+2}^2 + \cdots + \frac{p_{k-1}}{u_{k-1}u_k}a_k^2 = \\
&= \left(\frac{1}{u_{\nu+1}} - \frac{1}{u_\nu}\right)a_{\nu+1}^2 + \left(\frac{1}{u_{\nu+2}} - \frac{1}{u_{\nu+1}}\right)a_{\nu+2}^2 + \cdots + \left(\frac{1}{u_k} - \frac{1}{u_{k-1}}\right)a_k^2 = \\
&= -\frac{a_{\nu+1}^2}{u_\nu} + \frac{a_{\nu+1}^2 - a_{\nu+2}^2}{u_{\nu+1}} + \cdots + \frac{a_{k-1}^2 - a_k^2}{u_{k-1}} - \frac{a_k^2}{u_k} = -\frac{a_{\nu+1}^2}{u_\nu} + a_{\nu+1} + \\
&\quad 2(a_{\nu+2} + \cdots + a_k) = -\frac{a_\nu^2}{u_\nu} + a_\nu + 2(a_{\nu+1} + \cdots + a_k) = c_\nu^2 - \frac{a_\nu^2}{u_\nu}.
\end{aligned}
$$

Here, $c_\nu^2 = a_\nu + 2(a_{\nu+1} + \cdots + a_k) = p_\nu + 2^2p_{\nu+1} + \cdots + (k-\nu+1)^2p_k$.
Thus is our case $a_1 = a = MX_1$, $\sigma_1^2 = \sigma^2 = DX_1$. Besides we will note that $\sigma_{k-1}^2 = p_{k-1}p_k$.
Therefore, we have

$$\chi^2 = \sum_{\nu=1}^{k-1} y_\nu^2, \quad \sum_{\nu=1}^{k-1} c_\nu y_\nu = \tau - at.$$

Let's apply a lemma 1, as a result we will receve

$$\chi^2 = \sum_{\nu=1}^{k-2} z_\nu^2 + \frac{(\tau - at)^2}{\sigma^2}$$

where communication between y_ν and z_j is istablished by transformation

$$\begin{aligned}
y_1 &= \sqrt{\frac{p_1}{u_1 u_2}}\frac{a_2}{\sigma_1}\left(\frac{\tau - at}{\sigma_1}\right) + \sqrt{\frac{u_1}{u_2}}\frac{\sigma_2}{\sigma_1} z_1 \\
y_2 &= \sqrt{\frac{p_2}{u_2 u_3}}\frac{a_3}{\sigma_1}\left(\frac{\tau - at}{\sigma_1}\right) - \sqrt{\frac{p_1 p_2}{u_2 u_3}}\frac{a_2 a_3}{\sigma_1 \sigma_2} z_1 + \sqrt{\frac{u_2}{u_3}}\frac{\sigma_3}{\sigma_2} z_2 \\
&- \\
y_{k-2} &= \sqrt{\frac{p_{k-2}}{u_{k-2} u_{k-1}}}\frac{a_{k-2}}{\sigma_1}\left(\frac{\tau - at}{\sigma_1}\right) - \sqrt{\frac{p_1 p_{k-2}}{u_{k-2} u_{k-1}}}\frac{a_2 a_{k-1}}{\sigma_1 \sigma_2} z_1 - \cdots - \\
&- \sqrt{\frac{p_{k-3} p_{k-2}}{u_{k-2} u_{k-1}}}\frac{a_{k-2} a_{k-1}}{\sigma_{k-3} \sigma_{k-2}} z_{k-3} + \sqrt{\frac{u_{k-2}}{u_{k-1}}}\frac{\sigma_{k-1}}{\sigma_{k-2}} z_{k-2} \\
y_{k-1} &= \sqrt{\frac{p_{k-1}}{u_{k-1} u_k}}\frac{a_k}{\sigma_1}\left(\frac{\tau - at}{\sigma_1}\right) - \sqrt{\frac{p_1 p_{k-1}}{u_{k-1} u_k}}\frac{a_2 a_k}{\sigma_1 \sigma_2} z_1 - \cdots - \\
&- \sqrt{\frac{p_{k-2} p_{k-1}}{u_{k-1} u_k}}\frac{a_{k-1} a_k}{\sigma_{k-2} \sigma_{k-1}} z_{k-2}
\end{aligned} \tag{42}$$

On the basis of (40) and (42) we can write down the following transformation

$$\begin{aligned}
\frac{x_1}{\sqrt{p_1}} &= t + \frac{1-a}{\sigma_1}\tau + \frac{\sigma_2}{\sigma_1 \sqrt{p_1}} z_1 \\
\frac{x_2}{\sqrt{p_2}} &= t + \frac{2-a}{\sigma_1}\tau + \frac{\sqrt{p_1}}{\sigma_1 \sigma_2}(a_2 - c_2) z_1 + \frac{\sigma_3}{\sigma_2 \sqrt{p_2}} z_2 \\
&- \\
\frac{x_{k-2}}{\sqrt{p_{k-2}}} &= t + \frac{k-2-a}{\sigma_1}\tau + \frac{\sqrt{p_1}}{\sigma_1 \sigma_2}((k-3)a_2 - c_2) z_1 + \frac{\sqrt{p_2}}{\sigma_2 \sigma_3}((k-4)a_3 - c_3) z_2 + \cdots + \\
&\frac{\sqrt{p_{k-3}}}{\sigma_{k-3}\sigma_{k-2}}(a_{k-2} - c_{k-2}) z_{k-3} + \frac{\sigma_{k-1}}{\sigma_{k-2}\sqrt{p_{k-2}}} z_{k-2} \\
\frac{x_{k-1}}{\sqrt{p_{k-1}}} &= t + \frac{k-1-a}{\sigma_1}\tau + \frac{\sqrt{p_1}}{\sigma_1 \sigma_2}((k-2)a_2 - c_2) z_1 + \frac{\sqrt{p_2}}{\sigma_2 \sigma_3}((k-3)a_3 - c_3) z_2 + \cdots + \\
&\frac{\sqrt{p_{k-3}}}{\sigma_{k-3}\sigma_{k-2}}(2a_{k-2} - c_{k-2}) z_{k-3} + \frac{\sqrt{p_{k-2}}}{\sigma_{k-2}\sigma_{k-1}}(a_{k-1} - c_{k-1}) z_{k-2} \\
\frac{x_k}{\sqrt{p_k}} &= t + \frac{k-a}{\sigma_1}\tau + \frac{\sqrt{p_1}}{\sigma_1 \sigma_2}((k-1)a_2 - c_2) z_1 + \frac{\sqrt{p_2}}{\sigma_2 \sigma_3}((k-2)a_3 - c_3) z_2 + \cdots + \\
&+ \frac{\sqrt{p_{k-3}}}{\sigma_{k-3}\sigma_{k-2}}(3a_{k-2} - c_{k-2}) z_{k-3} + \frac{\sqrt{p_{k-2}}}{\sigma_{k-2}\sigma_{k-1}}(a_{k-1} - c_{k-1}) z_{k-2} + \\
&+ \frac{\sqrt{p_{k-2}}}{\sigma_{k-2}\sigma_{k-1}}(2a_{k-1} - c_{k-1}) z_{k-2}
\end{aligned} \tag{43}$$

As a results, we have

$$P(S_n = N) = \sum \frac{1}{(\sqrt{2\pi n})^{k-1}\sqrt{p_1 \dots p_k}} \exp\left\{-\frac{\tau^2}{2\sigma^2} - \frac{1}{2}\sum_{j=1}^{k-2} z_j^2\right\}\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

Furthe

$$\Delta z_1 \dots \Delta z_{k-2} = \frac{\sigma_1}{\sigma_2}\frac{1}{\sqrt{n^{k-2}p_1 \dots p_{k-2}}}.$$

Considering that

$$\frac{1}{(\sqrt{n})^{k-1}\sqrt{p_1 \dots p_k}} = \frac{1}{\sqrt{n\sigma_1^2}}\Delta z_1 \dots \Delta z_{k-2}$$

As a result, we will receive

$$P(S_n = N) = \frac{1}{\sqrt{2\pi n}\sigma} e^{-\frac{(N-na)^2}{2n\sigma^2}}\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right).$$

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