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Non-commutative Clarkson Inequalities for Symmetric Space Norm of $\tau-$Measurable Operators

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Abstract

We proved that if $(\mathcal{M},\tau)$ is a semi-finite von Neumann algebra, x and y are $\tau-$measurable operators, E is exact interpolation space for the couple $(L_1(0,\infty), L_\infty(0,\infty))$ and f is a increasing continuous function on $[0,\infty)$. Then

(i) in the case $f(0)=0$ and $g(t)=f(\sqrt{t})$ is operator convex,

$$\begin{aligned}\|f(|x|)+f(|y|)\|_{E(\mathcal{M})} &\le \|f(|x+y|)+f(|x-y|)\|_{E(\mathcal{M})}\\ &\le \|f(2|x|)+f(2|y|)\|_{E(\mathcal{M})}.\end{aligned}$$

(ii) in the case $h(t)=f(\sqrt{t})$ is concave,

$$\begin{aligned}\tfrac{1}{8}\|f(2|x|)+f(2|y|)\|_{E(\mathcal{M})} &\le \|f(|x+y|)+f(|x-y|)\|_{E(\mathcal{M})}\\ &\le 8\|f(|x|)+f(|y|)\|_{E(\mathcal{M})}.\end{aligned}$$

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1 Introduction

Let $\mathcal{B}(\mathcal{H})$ be the space of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. A unitarily invariant norm, denote by $|||\cdot|||$, is a norm defined on a norm ideal $C_{|||\cdot|||}$ in $\mathcal{B}(\mathcal{H})$, satisfying the property that $|||UAV||| = |||A|||$ for all operators $A \in C_{|||\cdot|||}$ and all unitary operators $U, V \in \mathcal{B}(\mathcal{H})$. With the exception of the usual operator norm, which is defined on all of $\mathcal{B}(\mathcal{H})$, each unitarily invariant norm $|||\cdot|||$ is a symmetric gauge function of the singular values, and $C_{|||\cdot|||}$ is a Banach space contained in the ideal of compact operators.

Hirzallah and Kittaneh in [11] proved non-commutative Clarkson inequalities for unitarily invariant norms: Let $A, B \in \mathcal{B}(\mathcal{H})$, $|||\cdot|||$ be unitarily invariant norm.

(i) If f is an increasing function on $[0, \infty)$ such that $f(0) = 0$, $\lim_{t\to\infty} f(t) = \infty$, and the inverse function of $g(t) = f(\sqrt{t})$ is operator monotone. Then

$$2|||f(|A|)+f(|B|)||| \leq |||f(|A+B|)+f(|A-B|)||| \leq 2^{-1}|||f(2|A|)+f(2|B|)|||. \tag{1}$$

(ii) If f is a nonnegative function on $[0, \infty)$ such that $h(t) = f(\sqrt{t})$ is operator monotone. Then

$$2^{-1}|||f(2|A|)+f(2|B|)||| \leq |||f(|A+B|)+f(|A-B|)||| \leq 2|||f(|A|)+f(|B|)|||. \tag{2}$$

The main result of this paper is to give similar inequalities in the case noncommutative symmetric space norm and $\tau-$ measurable operators.

2 Preliminaries

Throughout this paper, we denote by $\mathcal{M}$ a semi-finite von Neumann algebra in the Hilbert space $\mathcal{H}$ with a normal faithful semi-finite trace τ. The closed densely defined linear operator x in $\mathcal{H}$ with domain $D(x)$ is said to be affiliated with $\mathcal{M}$ if and only if $u^*xu = x$ for all unitary u which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$, the x said to be τ-measurable if for every $\varepsilon > 0$ there exists a projection $e \in \mathcal{M}$ such that $e(\mathcal{M}) \sqsubseteq D(x)$ and $\tau(e^\perp) < \varepsilon$ (where for any projection e we let $e^\perp = 1 - e$). The set of all τ-measure operators will be denoted by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a $*$-algebra with sum and product being the respective closure of the algebraic sum and product. Let $\mathcal{P}(\mathcal{M}$ be the lattice of projections of $\mathcal{M}$. The sets

$$\mathcal{N}(\varepsilon, \delta) = \{x \in L_0(\mathcal{M}) : \exists\ e \in \mathcal{P}(\mathcal{M}) \textit{ such that } \|xe\| < \varepsilon \textit{ and } \tau(e^\perp) < \delta\}$$

$(\varepsilon, \delta > 0)$ from a base at 0 for an metrizable Hausdorff topology in $L_0(\mathcal{M})$ called the measure topology. Equipped with the measure topology, $L_0(\mathcal{M})$ is a complete topological $*$-algebra (see [16]). For $x \in L_0(\mathcal{M})$, the generalized singular value function $\mu_{.}(x)$ of x is defined by

$$\mu_t(x) = \inf\{\|xe\|;\ e \in \mathcal{P}(\mathcal{M})\ \ \tau(e^{\perp}) \leq t\},\ t \geq 0.$$

$\mu_t(x)$ admits the following "minimax" representation:

$$\mu_t(x) = \inf_{\substack{e \in \mathcal{P}(\mathcal{M}) \\ \tau(1-e) \leq t}} [\sup_{\substack{\xi \in e(H) \\ \|\xi\| = 1}} \|x\xi\|]. \tag{3}$$

As shown shortly, we have $\mu_t(x) = \mu_t(x^{\frac{1}{2}})^2$ when x is positive.
Therefore, for a positive x, this expression reads

$$\mu_t(x) = \inf_{\substack{e \in \mathcal{P}(\mathcal{M}) \\ \tau(1-e) \leq t}} [\sup_{\substack{\xi \in e(H) \\ \|\xi\| = 1}} (x\xi, \xi)]. \tag{4}$$

We recall some terminology from the theory of rearrangement invariant space. Let $L_0([0,\infty))$ be the linear space of almost everywhere finite complex-valued Lebesgue measurable functions on $[0,\infty)$. For $f \in L_0([0,\infty))$, the right-continuous equimeasurable non-increasing rearrangement $\delta(f)$ of $|f|$ is defined by

$$\delta_t(f) = \inf\{r \in \mathrm{R} : d_{|f|}(r) \leq t\}, \quad t \in [0,\infty),$$

where $d_{|f|}$ is the distribution function of $|f|$ defined via

$$d_{|f|}(r) = |\{s : |f(s)| > r\}|, \quad r \in [0,\infty).$$

If $f, g \in L_0([0,\infty))$, then we say that f is submajorized by g and write $f \preceq g$ if and only if

$$\int_0^a \delta_t(f)dt \leq \int_0^a \delta_t(g)dt, \quad a \geq 0.$$

A Banach space $E \subset L_0([0,\infty))$ will be called,

(i) rearrangement invariant if and only if $f \in E, g \in L_0([0,\infty))$ and $\delta(f) \leq \delta(g)$ imply that $g \in E$ and $\|f\|_E \leq \|g\|_E$.

(ii) symmetric if and only if $f, g \in E$ and $\delta(f) \leq \delta(g)$ imply that $\|f\|_E \leq \|g\|_E$.

If E is a rearrangement invariant symmetric Banach function space on $[0,\infty)$, we define $E(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \mu(x) \in E\}$, and set $\|x\|_{E(\mathcal{M})} = \|\mu(x)\|_E$, $x \in E(\mathcal{M})$. Then $E(\mathcal{M})$ is a Banach space(see [6, 7]). It is called a noncommutative symmetric space associated with the rearrangement invariant symmetric Banach function space E and semi-finite von Neumann algebra $\mathcal{M}$.

Let

$$\mathbf{M}_2(\mathcal{M}) = \left\{ \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix},\ x_{i,j} \in \mathcal{M},\ i,j = 1,2 \right\},$$

then $\mathbf{M}_2(\mathcal{M})$ is a von Neumann algebra on the Hilbert space $\mathcal{H}\oplus\mathcal{H}$ with trace

$$\sigma(x)=\sigma(\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2}\end{pmatrix})=\sum_{k=1}^{2}\tau(x_{k,k}).$$

For $x\in\mathbf{M}_2(\mathcal{M})$, let

$$\mathcal{C}(x)=\mathcal{C}(\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2}\end{pmatrix})=\begin{pmatrix} x_{1,1} & 0 \\ 0 & x_{2,2}\end{pmatrix}.$$

Then we have that

$$\mathcal{C}(x)=\frac{1}{2}\sum_{j=1}^{2}(u^*)^j x u^j, \tag{5}$$

where $u=(e_1-e_2),\ e_1=\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix},\ e_2=\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}$ and 1 is the unit operator of $\mathcal{M}$(see [3]).

Notice that the sub von Neumann algebra $e_1\mathbf{M}_2(\mathcal{M})e_1=\left\{\begin{pmatrix} x & 0 \\ 0 & 0\end{pmatrix},\ x\in\mathcal{M}\right\}$ of $\mathbf{M}_2(\mathcal{M})$ is isomorphic to von Neumann algebra $\mathcal{M}$, where e_1 as in the previous paragraph. Define $T:\ L_0(e_1\mathbf{M}_2(\mathcal{M})e_1)\to L_0(\mathcal{M})$ by $T\begin{pmatrix} x & 0 \\ 0 & 0\end{pmatrix}=x$. It is clear that $\|Tx\|_{L_1(\mathbf{M}_2(\mathcal{M}))}=\|x\|_{L_1(\mathcal{M})}$ and $\|Tx\|=\|x\|$. If E is a noncommutative symmetric space and exact interpolation space for the couple $(L_1(0,\infty),L_\infty(0,\infty))$. Then by Theorem 3.4 in [6], we have that $\|Tx\|_{E(\mathbf{M}_2(\mathcal{M}))}=\|x\|_{E(\mathcal{M})}$, i.e.

$$\|\begin{pmatrix} x & 0 \\ 0 & 0\end{pmatrix}\|_{E(\mathbf{M}_2(\mathcal{M}))}=\|x\|_{E(\mathcal{M})} \tag{6}$$

3 Main results

To achieve our goal , we need the following lemmas

Lemma 3.1 *Let x be a positive $\tau-$measurable operator.*
(i) If f is a convex function on $[0,\infty)$, then

$$f(\langle x\xi,\xi\rangle)\le\langle f(x)\xi,\xi\rangle \tag{7}$$

for every unit vector ξ in $D(x)$.
(ii) If g is a concave function on $[0,\infty)$, then

$$\langle g(x)\xi,\xi\rangle\le g(\langle x\xi,\xi\rangle) \tag{8}$$

for every unit vector ξ in $D(x)$.

Proof. We prove only (i). The proof of (ii) is similar. Let $x = \int_0^\infty \lambda de_\lambda(x)$ be spectral decomposition of x, then
$f(x) = \int_0^\infty f(\lambda) de_\lambda(x)$. Since $\int_0^\infty d\langle e_\lambda(x)\xi, \xi\rangle = 1$ and $\langle x\xi, \xi\rangle = \int_0^\infty \lambda d\langle e_\lambda(x)\xi, \xi\rangle$, for every unit vector ξ in $D(x)$. By Jensen's inequality, we obtain that

$$f(\langle x\xi, \xi\rangle) = f(\int_0^\infty \lambda d\langle e_\lambda(x)\xi, \xi\rangle) \leq \int_0^\infty f(\lambda) d\langle e_\lambda(x)\xi, \xi\rangle = \langle f(x)\xi, \xi\rangle.$$

Lemma 3.2 *Let x and y be positive τ-measurable operators and let E be an exact interpolation space for the couple $(L_1(0,\infty), L_\infty(0,\infty))$.*

(i) If f is a non-negative operator convex function on $[0,\infty)$ with $f(0) = 0$, then

$$\|f(x) + f(y)\|_{E(\mathcal{M})} \leq 2\|f(x+y)\|_{E(\mathcal{M})}. \tag{9}$$

(ii) If g is a non-negative increasing continuous concave function on $[0,\infty)$, then

$$\|g(x+y)\|_{E(\mathcal{M})} \leq 4\|g(x) + g(y)\|_{E(\mathcal{M})}, \tag{10}$$

Proof. Let $z = \begin{pmatrix} x^{\frac{1}{2}} & y^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}$, then $zz^* = \begin{pmatrix} x+y & 0 \\ 0 & 0 \end{pmatrix}$ and $z^*z = \begin{pmatrix} x & x^{\frac{1}{2}}y^{\frac{1}{2}} \\ y^{\frac{1}{2}}x^{\frac{1}{2}} & y \end{pmatrix}$.
(i) Since f is operator convex, by (5) we have that

$$\begin{aligned} \| \begin{pmatrix} f(x) & 0 \\ 0 & f(y) \end{pmatrix} \|_{E(\mathbf{M}_2(\mathcal{M}))} &= \|f(\mathcal{C}(z^*z))\|_{E(\mathbf{M}_2(\mathcal{M}))} = \|\mathcal{C}(f(z^*z))\|_{E(\mathbf{M}_2(\mathcal{M}))} \\ &\leq \|f(z^*z)\|_{E(\mathbf{M}_2(\mathcal{M}))} = \|f(zz^*)\|_{E(\mathbf{M}_2(\mathcal{M}))} \\ &= \| \begin{pmatrix} f(x+y) & 0 \\ 0 & 0 \end{pmatrix} \|_{E(\mathbf{M}_2(\mathcal{M}))}. \end{aligned}$$

On the other hand, $\begin{pmatrix} f(x)+f(y) & 0 \\ 0 & f(x)+f(y) \end{pmatrix} = \begin{pmatrix} f(x) & 0 \\ 0 & f(y) \end{pmatrix} + u \begin{pmatrix} f(x) & 0 \\ 0 & f(y) \end{pmatrix} u^*$, where $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is unitary. So

$$\| \begin{pmatrix} f(x)+f(y) & 0 \\ 0 & f(x)+f(y) \end{pmatrix} \|_{E(\mathbf{M}_2(\mathcal{M}))} \leq 2\| \begin{pmatrix} f(x) & 0 \\ 0 & f(y) \end{pmatrix} \|_{E(\mathbf{M}_2(\mathcal{M}))}.$$

Hence

$$\| \begin{pmatrix} f(x)+f(y) & 0 \\ 0 & f(x)+f(y) \end{pmatrix} \|_{E(\mathbf{M}_2(\mathcal{M}))} \leq 2\| \begin{pmatrix} f(x+y) & 0 \\ 0 & 0 \end{pmatrix} \|_{E(\mathbf{M}_2(\mathcal{M}))}.$$

We using $\begin{pmatrix} f(x)+f(y) & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} f(x)+f(y) & 0 \\ 0 & f(x)+f(y) \end{pmatrix}$ and (6), to obtain (9).
(ii) By Lemma 2.5 in [8], we have that

$$\begin{aligned} \mu_t(g(x+y)) &= g(\mu_t(x+y)) \leq g(\mu_{\frac{t}{2}}(x) + \mu_{\frac{t}{2}}(y)) \leq g(\mu_{\frac{t}{2}}(x)) + g(\mu_{\frac{t}{2}}(y)) \\ &= \mu_{\frac{t}{2}}(g(x)) + \mu_{\frac{t}{2}}(g(y)) \leq 2\mu_{\frac{t}{2}}(g(x)+g(y)). \end{aligned}$$

Since $\|2\mu_{\frac{t}{2}}(g(x)+g(y))\|_{L_1(0,\infty)} \leq 4\|\mu_t(g(x)+g(y))\|_{L_1(0,\infty)}$, $\|2\mu_{\frac{t}{2}}(g(x)+g(y))\|_{L_\infty(0,\infty)} \leq 2\|\mu_t(g(x)+g(y))\|_{L_\infty(0,\infty)}$ and E is an exact interpolation space for the couple $(L_1(0,\infty), L_\infty(0,\infty))$. So

$$\begin{aligned}\|g(x+y)\|_{E(\mathcal{M})} &= \|g(\mu_t(x+y))\|_E \leq \|2\mu_{\frac{t}{2}}(g(x)+g(y))\|_E \\ &= 4\|\mu_t(g(x)+g(y))\|_E = 4\|g(x)+g(y)\|_{E(\mathcal{M})}.\end{aligned}$$

Theorem 3.3 *Let x and y be τ measurable operators and let f be an increasing continuous function on $[0,\infty)$ such that $f(0)=0$ and $g(t)=f(\sqrt{t})$ is operator convex. Then*

$$\begin{aligned}\|f(|x|)+f(|y|)\|_{E(\mathcal{M})} &\leq \|f(|x+y|)+f(|x-y|)\|_{E(\mathcal{M})} \\ &\leq \|f(2|x|)+f(2|y|)\|_{E(\mathcal{M})}.\end{aligned} \tag{11}$$

Proof. Since g is convex, by Lemma 3.1(i) we have that

$$\begin{aligned}\langle (f(|x+y|)+f(|x-y|))\xi,\xi\rangle &= \langle (g(|x+y|^2))\xi,\xi\rangle + \langle (g(|x-y|^2))\xi,\xi\rangle \\ &\geq g(\langle |x+y|^2\xi,\xi\rangle) + g(\langle |x-y|^2\xi,\xi\rangle) \\ &\geq 2g(\tfrac{\langle |x+y|^2\xi,\xi\rangle+\langle |x-y|^2\xi,\xi\rangle}{2}) \\ &= 2g(\langle (|x|^2+|y|^2)\xi,\xi\rangle),\end{aligned}$$

for any unit vector ξ in $D(x)\cap D(y)$. Using the equation (4) and the fact that g is increasing, we see that

$$\begin{aligned}\mu_t[(f(|x+y|) &+ f(|x-y|))] \\ &= \inf_{\substack{e\in\mathcal{P}(\mathcal{M})\\ \tau(1-e)\leq t}} [\sup_{\substack{\xi\in e(H)\\ \|\xi\|=1}} \langle (f(|x+y|)+f(|x-y|))\xi,\xi\rangle] \\ &\geq \inf_{\substack{e\in\mathcal{P}(\mathcal{M})\\ \tau(1-e)\leq t}} [\sup_{\substack{\xi\in E(H)\\ \|\xi\|=1}} 2g(\langle (|x|^2+|y|^2)\xi,\xi\rangle)] \\ &= 2g[\inf_{\substack{e\in\mathcal{P}(\mathcal{M})\\ \tau(1-e)\leq t}} (\sup_{\substack{\xi\in E(H)\\ \|\xi\|=1}} \langle (|x|^2+|y|^2)\xi,\xi\rangle)] \\ &= 2g[\mu_t(|x|^2+|y|^2)] = 2\mu_t[g(|x|^2+|y|^2)].\end{aligned}$$

By Lemma 3.2(i), it follows that

$$\begin{aligned}\|f(|x+y|)+f(|x-y|)\|_{E(\mathcal{M})} &\geq 2\|g(|x|^2+|y|^2)\|_{E(\mathcal{M})} \\ &\geq \|g(|x|^2)+g(|y|^2)\|_{E(\mathcal{M})} \\ &= \|f(|x|)+f(|y|)\|_{E(\mathcal{M})}.\end{aligned}$$

This is the first inequality in (11). We using this inequality to obtain that

$$\begin{aligned}\|f(2|x|) &+ f(2|y|)\|_{E(\mathcal{M})} \\ &= \|f(|(x+y)+(x-y)|)+f(|(x+y)-(x-y)|)\|_{E(\mathcal{M})} \\ &\geq \|f(|x+y|)+f(|x-y|)\|_{E(\mathcal{M})}.\end{aligned}$$

So obtain the second inequality in (11).

Theorem 3.4 *Let x and y be τ measurable operators and let f be a non-negative increasing continuous function on $[0,\infty)$ such that $h(t) = f(\sqrt{t})$ is concave. Then*

$$\begin{aligned} \tfrac{1}{8}\|f(2|x|)+f(2|y|)\|_{E(\mathcal{M})} &\leq \|f(|x+y|)+f(|x-y|)\|_{E(\mathcal{M})} \\ &\leq 8\|f(|x|)+f(|y|)\|_{E(\mathcal{M})}. \end{aligned} \tag{12}$$

Proof. Since h is concave. Now for any unit vector ξ in $D(x)\cap D(y)$, by Lemma 3.1(ii) we have that

$$\begin{aligned} \langle (f(|x+y|)+f(|x-y|))\xi,\xi\rangle &= \langle (h(|x+y|^2))\xi,\xi\rangle + \langle (h(|x-y|^2))\xi,\xi\rangle \\ &\leq h(\langle |x+y|^2\xi,\xi\rangle) + h(\langle |x-y|^2\xi,\xi\rangle) \\ &\leq 2h(\tfrac{\langle |x+y|^2\xi,\xi\rangle + \langle |x-y|^2\xi,\xi\rangle}{2}) \\ &= 2h(\langle (|x|^2+|y|^2)\xi,\xi\rangle). \end{aligned}$$

Using the equation (4) and the fact that h is monotone, we obtain that

$$\begin{aligned} \mu_t[(f(|x+y|) &+f(|x-y|))] \\ &= \inf_{\substack{e\in\mathcal{P}(\mathcal{M}) \\ \tau(1-e)\leq t}} [\sup_{\substack{\xi\in e(H) \\ \|\xi\|=1}} \langle (f(|x+y|)+f(|x-y|))\xi,\xi\rangle] \\ &\leq \inf_{\substack{e\in\mathcal{P}(\mathcal{M}) \\ \tau(1-e)\leq t}} [\sup_{\substack{\xi\in e(H) \\ \|\xi\|=1}} 2h(\langle (|x|^2+|y|^2)\xi,\xi\rangle)] \\ &= 2h[\inf_{\substack{e\in\mathcal{P}(\mathcal{M}) \\ \tau(1-e)\leq t}} (\sup_{\substack{\xi\in e(H) \\ \|\xi\|=1}} \langle (|x|^2+|y|^2)\xi,\xi\rangle)] \\ &= 2h[\mu_t(|x|^2+|y|^2)] = 2\mu_t[h(|x|^2+|y|^2)]. \end{aligned}$$

Hence, by Lemma 3.2(ii),

$$\begin{aligned} \|f(|x+y|)+f(|x-y|)\|_{E(\mathcal{M})} &\leq 2\|h(|x|^2+|y|^2)\|_{E(\mathcal{M})} \\ &\leq 8\|h(|x|^2)+h(|y|^2)\|_{E(\mathcal{M})} \\ &= 8\|f(|x|)+f(|y|)\|_{E(\mathcal{M})}. \end{aligned}$$

$$\begin{aligned} \|f(2|x|) &+f(2|y|)\|_{E(\mathcal{M})} \\ &= \|f(|(x+y)+(x-y)|)+f(|(x+y)-(x-y)|)\|_{E(\mathcal{M})} \\ &\leq 8\|f(|x+y|)+f(|x-y|)\|_{E(\mathcal{M})}. \end{aligned}$$

Specializing Theorems 3.3 and 3.4 to the functions $f(t)=t^p$ $(2\leq p\leq 4)$ and $f(t)=t^p$ $(0<p\leq 2)$, respectively, we obtain the following.

Corollary 3.5 *Let x and y be τ measurable operators. Then*

$$\|\,|x|^p+|y|^p\|_{E(\mathcal{M})} \leq \|\,|x+y|^p+|x-y|^p\|_{E(\mathcal{M})} \leq 2^p\|\,|x|^p+|y|^p\|_{E(\mathcal{M})}. \tag{13}$$

for $2\leq p\leq 4$, and

$$2^{p-3}\|\,|x|^p+|y|^p\|_{E(\mathcal{M})} \leq \|\,|x+y|^p+|x-y|^p\|_{E(\mathcal{M})} \leq 8\|\,|x|^p+|y|^p\|_{E(\mathcal{M})}. \tag{14}$$

for $0\leq p\leq 2$.

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